Mathematics towards industry Weekend

Quantum graphs
in
Mathematics, Physics and Applications
QGRAPH Network meeting

Graph-like models for bent waveguides:
the role of the boundary conditions

Claudio Cacciapuoti

Hausdorff Center for Mathematics, Bonn Universität
Summary

• Introduction: Metric Graphs & Constrained Systems

• Waveguides & Graphs

• Remarks & References
Summary

- Introduction: Metric Graphs & Constrained Systems
- Waveguides & Graphs
- Remarks & References
Summary

• Introduction: Metric Graphs & Constrained Systems

• Waveguides & Graphs

• Remarks & References
• Introduction: Metric Graphs & Constrained Systems

• Waveguides & Graphs

• Remarks & References
A metric graph is realized by a set of segments (edges) \( \{e_j\}_{j=1}^n \) and a set of points (vertices) \( \{v_j\}_{j=1}^m \).
• A metric graph is realized by a set of segments (edges) \( \{e_j\}_{j=1}^{n} \) and a set of points (vertices) \( \{v_j\}_{j=1}^{m} \)

• Self-adjoint, (one-dimensional) differential operators can be defined on the graph
Introduction

• A metric graph is realized by a set of segments (edges) \( \{e_j\}_{j=1}^{n} \) and a set of points (vertices) \( \{v_j\}_{j=1}^{m} \).

• Self-adjoint, (one-dimensional) differential operators can be defined on the graph.

• Graphs as one-dimensional approximations for constrained dynamics in which “transverse dimensions are small with respect to longitudinal ones.”
Hilbert Space

\[ \mathcal{H} = \bigoplus_{j=1}^{n} L^2((0, l_j)) \quad l_j \in (0, +\infty) ; \quad f = (f_1, \ldots, f_n) \in \mathcal{H} \]
• Hilbert Space

\[ \mathcal{H} = \bigoplus_{j=1}^{n} L^2((0, l_j)) \quad l_j \in (0, +\infty) \quad f = (f_1, \ldots, f_n) \in \mathcal{H} \]

• Laplacian on \( G \)

\[ -\Delta_g f = \left( -\frac{d^2 f_1}{ds^2}, \ldots, -\frac{d^2 f_n}{ds^2} \right) \]
• Hilbert Space

\[ \mathcal{H} = \bigoplus_{j=1}^{n} L^2((0, l_j)) \quad l_j \in (0, +\infty) ; \quad f = (f_1, \ldots, f_n) \in \mathcal{H} \]

• Laplacian on \( \mathcal{G} \)

\[ -\Delta_g f = \left( -\frac{d^2 f_1}{ds^2}, \ldots, -\frac{d^2 f_n}{ds^2} \right) \]

\[ \mathcal{D}(-\Delta_g) = \bigoplus_{j=1}^{n} H^2((0, l_j)) + \text{self-adjoint conditions in the vertices and endpoints} \]
Kostrykin and Schrader '99: $-\Delta_g^{AB}$

\[
\begin{pmatrix}
  f_1(v) \\
  \vdots \\
  f_n(v)
\end{pmatrix}
A
\begin{pmatrix}
  f_1'(v) \\
  \vdots \\
  f_n'(v)
\end{pmatrix}
= B
\begin{pmatrix}
  f_1'(v) \\
  \vdots \\
  f_n'(v)
\end{pmatrix}
\]

- $A$ and $B$ are $n \times n$ matrices
- $(A|B)$ has maximal rank $n$
- $AB^*$ is self-adjoint
• Kostrykin and Schrader '99: $-\Delta_G^{AB}$

$$A \begin{pmatrix} f_1(v) \\ \vdots \\ f_n(v) \end{pmatrix} = B \begin{pmatrix} f'_1(v) \\ \vdots \\ f'_n(v) \end{pmatrix}$$

- $A$ and $B$ are $n \times n$ matrices
- $(A|B)$ has maximal rank $n$
- $AB^*$ is self-adjoint

• Decoupling condition:

$$f_1(v) = f_2(v) = \cdots = f_n(v) = 0$$
• Kostrykin and Schrader ’99: $-\Delta_g^{AB}$

\[
A \begin{pmatrix} f_1(v) \\ \vdots \\ f_n(v) \end{pmatrix} = B \begin{pmatrix} f_1'(v) \\ \vdots \\ f_n'(v) \end{pmatrix}
\]

- $A$ and $B$ are $n \times n$ matrices
- $(A|B)$ has maximal rank $n$
- $AB^*$ is self-adjoint

• Decoupling condition:

$f_1(v) = f_2(v) = \cdots = f_n(v) = 0$

• Standard condition:

$f_1(v) = f_2(v) = \cdots = f_n(v)$

\[
\sum_{j=1}^{n} f_j'(v) = 0
\]
Introduction

• Define $-\Delta\Omega$ (Boundary Conditions on $\partial\Omega$)
• Analyze the "convergence" of $-\Delta\Omega$ to $-\Delta ABG$ as $\Omega$ "collapses" onto $G$
• Far away from the vertex region the dynamics is factorized
• Transverse problem: discrete spectrum, energy gap between the eigenvalues of order $1/d^2$
• The dynamics on the graph must refer to a fixed transverse mode
• Define $-\Delta_{\Omega}$ (Boundary Conditions on $\partial\Omega$)
• Define $-\Delta_{\Omega}$ (Boundary Conditions on $\partial \Omega$)
• Analyze the "convergence" of $-\Delta_{\Omega}$ to $-\Delta_{\mathcal{G}}^{AB}$ as $\Omega$ "collapses" onto $\mathcal{G}$
• Define $-\Delta_\Omega$ (Boundary Conditions on $\partial\Omega$)
• Analyze the “convergence” of $-\Delta_\Omega$ to $-\Delta_G^{AB}$ as $\Omega$ “collapses” onto $\mathcal{G}$
• Far away from the vertex region the dynamics is factorized
• Define $-\Delta_\Omega$ (Boundary Conditions on $\partial\Omega$)
• Analyze the “convergence” of $-\Delta_\Omega$ to $-\Delta^{AB}_G$ as $\Omega$ “collapses” onto $G$
• Far away from the vertex region the dynamics is factorized
• Transverse problem: discrete spectrum, energy gap between the eigenvalues of order $1/d^2$
• Define $-\Delta_\Omega$ (Boundary Conditions on $\partial \Omega$)
• Analyze the “convergence” of $-\Delta_\Omega$ to $-\Delta^A_\mathcal{G}$ as $\Omega$ “collapses” onto $\mathcal{G}$
• Far away from the vertex region the dynamics is factorized
• Transverse problem: discrete spectrum, energy gap between the eigenvalues of order $1/d^2$
• The dynamics on the graph must refer to a fixed transverse mode
Summary

• Introduction: Metric Graphs & Constrained Systems

• Waveguides & Graphs

• Remarks & References
Waveguide Collapsing onto a Graph

\[
\begin{align*}
-\Delta_R \Omega - \Delta_R \Omega f &= -\Delta f \\
D(-\Delta_R \Omega) &= \psi \in H^2(\Omega) \\
\partial \psi \bigg|_{\partial \Omega} + \alpha \psi \bigg|_{\partial \Omega} &= 0 \\
\alpha &= 0, \text{ Acoustic and Electromagnetic Waveguides} \\
\alpha &= \pm \infty, \text{ Quantum Waveguides}
\end{align*}
\]
• (Symmetric) Robin Laplacian, $-\Delta^R_\Omega$

$$-\Delta^R_\Omega f = -\Delta f$$

with domain

$$\mathcal{D}(-\Delta^R_\Omega) = \left\{ \psi \in H^2(\Omega) \text{ s.t. } \frac{\partial \psi}{\partial n} \bigg|_{\partial \Omega} + \alpha \psi \bigg|_{\partial \Omega} = 0 \right\} ; \quad \alpha \in \mathbb{R}$$
Waveguide Collapsing onto a Graph

- (Symmetric) Robin Laplacian, $-\Delta^R_\Omega$

$$-\Delta^R_\Omega f = -\Delta f$$

with domain

$$\mathcal{D}(-\Delta^R_\Omega) = \left\{ \psi \in H^2(\Omega) \text{ s.t. } \frac{\partial \psi}{\partial n}\bigg|_{\partial\Omega} + \alpha \psi\bigg|_{\partial\Omega} = 0 \right\} ; \quad \alpha \in \mathbb{R}$$

- Neumann B.C.: $\alpha = 0$, Acoustic and Electromagnetic Waveguides
- Dirichlet B.C.: "$\alpha = +\infty$", Quantum Waveguides
The Waveguide $\Omega$

- The Waveguide $\Omega$

  - $C : \mathbb{R} \to \mathbb{R}^2$
  
    $C(s) := \{(\gamma_1(s), \gamma_2(s))| s \in \mathbb{R}\}$

    $\dot{\gamma}_1(s)^2 + \dot{\gamma}_2(s)^2 = 1$

  - $C$ has no self-intersections
The Waveguide $\Omega$

- The Waveguide $\Omega$
  
  - $C : \mathbb{R} \rightarrow \mathbb{R}^2$
    
    $$C(s) := \{(\gamma_1(s), \gamma_2(s)) | s \in \mathbb{R}\}$$
    
    $$\dot{\gamma}_1(s)^2 + \dot{\gamma}_2(s)^2 = 1$$
  
  - $C$ has no self-intersections

- Signed curvature:
  
  $$\gamma(s) := \dot{\gamma}_2(s)\ddot{\gamma}_1(s) - \dot{\gamma}_1(s)\ddot{\gamma}_2(s) ; \quad \gamma \in C_0^\infty(\mathbb{R})$$
The Waveguide $\Omega$

- The Waveguide $\Omega$
  - $C : \mathbb{R} \rightarrow \mathbb{R}^2$
    - $C(s) := \{(\gamma_1(s), \gamma_2(s)) | s \in \mathbb{R}\}$
    - $\dot{\gamma}_1(s)^2 + \dot{\gamma}_2(s)^2 = 1$
  - $C$ has no self-intersections

- Signed curvature:
  - $\gamma(s) := \dot{\gamma}_2(s)\ddot{\gamma}_1(s) - \dot{\gamma}_1(s)\ddot{\gamma}_2(s) ; \quad \gamma \in C^\infty_0(\mathbb{R})$

- $\Omega := \{(x, y) \in \mathbb{R}^2 \mid (x, y) = (\gamma_1(s), \gamma_2(s)) + u\hat{n}(s), \forall s \in \mathbb{R}, u \in (-1, 1)\} \subset \mathbb{R}^2$
  - $\hat{n}(s) = (-\dot{\gamma}_2, \dot{\gamma}_1)$ is the vector (of unit norm) orthogonal to $C$
  - $\Omega$ is a waveguide of constant width
  - Global System of Coordinates: $s \in \mathbb{R}, u \in (-1, 1)$
• Scaling of $\Omega$: $\varepsilon$ is a “small” dimensionless parameter

Scaling of the Width: $u \rightarrow \varepsilon a$

Scaling of the Curvature: $\gamma(s) \rightarrow \gamma(\varepsilon s) := 1/\varepsilon \gamma(s)$; $\theta = \int_{\mathbb{R}} \gamma(s) ds = \int_{\mathbb{R}} \gamma(\varepsilon s) ds = \theta/\varepsilon$

Scaling of the Boundary Constant: $\alpha \rightarrow \alpha/\varepsilon = \alpha \varepsilon$

Scaling of the Waveguide: $\Omega \rightarrow \Omega/\varepsilon$

Scaling of the Laplacian in the Waveguide: $-\Delta R_{\Omega} \rightarrow -\Delta R_{\Omega/\varepsilon}$

$Vol_v \sim \varepsilon \cdot \varepsilon^a$

d $\sim \varepsilon^a$
• Scaling of $\Omega$: $\varepsilon$ is a “small” dimensionless parameter

• Scaling of the Width: $u \rightarrow \varepsilon^a u$ \quad a > 3
Scaling

- Scaling of $\Omega$: $\varepsilon$ is a “small” dimensionless parameter

- Scaling of the Width: $u \rightarrow \varepsilon^a u$ for $a > 3$
- Scaling of the Curvature:

\[
\gamma(s) \rightarrow \gamma_\varepsilon(s) := \frac{1}{\varepsilon} \gamma \left( \frac{s}{\varepsilon} \right); \quad \theta = \int_{\mathbb{R}} \gamma(s) ds = \int_{\mathbb{R}} \gamma_\varepsilon(s) ds = \theta_\varepsilon
\]
• Scaling of $\Omega$: $\varepsilon$ is a “small” dimensionless parameter

\[ \text{Vol}_y \sim \varepsilon \cdot \varepsilon^a \]

\[ d \sim \varepsilon^a \]

• Scaling of the Width: $u \rightarrow \varepsilon^a u \quad a > 3$

• Scaling of the Curvature:

\[ \gamma(s) \rightarrow \gamma_\varepsilon(s) := \frac{1}{\varepsilon} \gamma\left(\frac{s}{\varepsilon}\right); \quad \theta = \int_\mathbb{R} \gamma(s) ds = \int_\mathbb{R} \gamma_\varepsilon(s) ds = \theta_\varepsilon \]

• Scaling of the Boundary Constant:

\[ \alpha \rightarrow \alpha_\varepsilon = \alpha/\varepsilon^a \]
Scaling

- Scaling of $\Omega$: $\varepsilon$ is a “small” dimensionless parameter

\[ \text{Vol}_\nu \sim \varepsilon \cdot \varepsilon^a \]

\[ d \sim \varepsilon^a \]

- Scaling of the Width: $u \rightarrow \varepsilon^a u \quad a > 3$
- Scaling of the Curvature:

\[ \gamma(s) \rightarrow \gamma_\varepsilon(s) := \frac{1}{\varepsilon} \gamma\left(\frac{s}{\varepsilon}\right) ; \quad \theta = \int_{\mathbb{R}} \gamma(s) ds = \int_{\mathbb{R}} \gamma_\varepsilon(s) ds = \theta_\varepsilon \]

- Scaling of the Boundary Constant:

\[ \alpha \rightarrow \alpha_\varepsilon = \alpha/\varepsilon^a \]

- Scaling of the Waveguide: $\Omega \rightarrow \Omega_\varepsilon$
• Scaling of $\Omega$: $\varepsilon$ is a “small” dimensionless parameter

$\text{Vol}_V \sim \varepsilon \cdot \varepsilon^a$

$\theta = \int_{\mathbb{R}} \gamma(s) ds = \int_{\mathbb{R}} \gamma_\varepsilon(s) ds = \theta_\varepsilon$

• Scaling of the Width: $u \rightarrow \varepsilon^a u \quad a > 3$

• Scaling of the Curvature:

$\gamma(s) \rightarrow \gamma_\varepsilon(s) = \frac{1}{\varepsilon} \gamma\left(\frac{s}{\varepsilon}\right)$

• Scaling of the Boundary Constant:

$\alpha \rightarrow \alpha_\varepsilon = \alpha / \varepsilon^a$

• Scaling of the Waveguide: $\Omega \rightarrow \Omega_\varepsilon$

• Scaling of the Laplacian in the Waveguide: $-\Delta^R_\Omega \rightarrow -\Delta^R_{\Omega_\varepsilon}$
Summary of the Proof

Step 1  Reduction to a One Dimensional Dynamics (dependent on $\varepsilon$)

Step 2  Analysis of the One Dimensional Problem (limit $\varepsilon \to 0$)
Summary of the Proof

Step 1  Reduction to a One Dimensional Dynamics (dependent on $\varepsilon$)

Step 2  Analysis of the One Dimensional Problem (limit $\varepsilon \to 0$)
Summary of the Proof

Step 1  Reduction to a One Dimensional Dynamics (dependent on $\varepsilon$)

Step 2  Analysis of the One Dimensional Problem (limit $\varepsilon \to 0$)
Summary of the Proof

Step 1 Reduction to a One Dimensional Dynamics (dependent on $\varepsilon$)

Step 2 Analysis of the One Dimensional Problem (limit $\varepsilon \to 0$)
Step 1: Reduced Hamiltonian

- The operator $-\Delta^R_{\Omega_{\varepsilon}}$ on $L^2(\Omega_{\varepsilon})$ is unitarily equivalent to $H_\varepsilon$ on $L^2(\mathbb{R} \times (-1, 1))$

\[
H_\varepsilon = -\frac{\partial}{\partial s} \left( \frac{1}{1 + \varepsilon^{a-1} u \gamma(s/\varepsilon)^2} \right) \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} + W_\varepsilon(s, u),
\]
Step 1: Reduced Hamiltonian

- The operator \(-\Delta_{\Omega_\varepsilon}^R\) on \(L^2(\Omega_\varepsilon)\) is unitarily equivalent to \(H_\varepsilon\) on \(L^2(\mathbb{R} \times (-1, 1))\)

\[
H_\varepsilon = -\frac{\partial}{\partial s} \frac{1}{(1 + \varepsilon^{a-1} u \gamma(s/\varepsilon))^2} \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} + W_\varepsilon(s, u),
\]

- The Boundary Conditions are not preserved
Step 1: Reduced Hamiltonian

- The operator \(-\Delta^R_{\Omega_\varepsilon}\) on \(L^2(\Omega_\varepsilon)\) is unitarily equivalent to \(H_\varepsilon\) on \(L^2(\mathbb{R} \times (-1, 1))\)

\[
H_\varepsilon = -\frac{\partial}{\partial s} \frac{1}{(1 + \varepsilon^{a-1}u\gamma(s/\varepsilon))^2} \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} + W_\varepsilon(s, u),
\]

- The Boundary Conditions are not preserved

\[
\psi \in \mathcal{D}(-\Delta^R_{\Omega_\varepsilon}) \ ; \quad \left\{ \frac{\partial \psi}{\partial \hat{n}} \bigg|_{\partial \Omega_\varepsilon} + \alpha \psi \bigg|_{\partial \Omega_\varepsilon} = 0 \right\}
\]
Step 1: Reduced Hamiltonian

• The operator $-\Delta^R_{\Omega_\varepsilon}$ on $L^2(\Omega_\varepsilon)$ is unitarily equivalent to $H_\varepsilon$ on $L^2(\mathbb{R} \times (-1, 1))$

$$H_\varepsilon = -\frac{\partial}{\partial s} \frac{1}{(1 + \varepsilon^{a-1} u \gamma(s/\varepsilon))^2} \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} + W_\varepsilon(s, u),$$

• The Boundary Conditions are not preserved

$$\psi \in \mathcal{D}(-\Delta^R_{\Omega_\varepsilon}) ; \quad \left\{ \frac{\partial \psi}{\partial \hat{n}} \right\}_{\partial \Omega_\varepsilon} + \alpha \psi \bigg|_{\partial \Omega_\varepsilon} = 0$$

$$\Downarrow$$

$$\tilde{\psi} = U\psi ; \quad \left\{ \pm \frac{\partial \tilde{\psi}}{\partial u} \bigg|_{u=\pm 1} + \left[ \alpha \mp \frac{\varepsilon^{a-1} \gamma(s/\varepsilon)}{2(1 \pm \varepsilon^{a-1} \gamma(s/\varepsilon))} \right] \tilde{\psi} \bigg|_{u=\pm 1} = 0 \right\} \quad (1)$$
Step 1: Reduced Hamiltonian

- The operator $-\Delta^R_{\Omega_\varepsilon}$ on $L^2(\Omega_\varepsilon)$ is unitarily equivalent to $H_\varepsilon$ on $L^2(\mathbb{R} \times (-1, 1))$

$$H_\varepsilon = -\frac{\partial}{\partial s} \frac{1}{(1 + \varepsilon^{a-1} u \gamma(s/\varepsilon))^2} \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} + W_\varepsilon(s, u),$$

- The Boundary Conditions are not preserved

$$\psi \in \mathcal{D}(-\Delta^R_{\Omega_\varepsilon}) ; \quad \left\{ \frac{\partial \psi}{\partial \hat{n}} \bigg|_{\partial \Omega_\varepsilon} + \alpha \psi \bigg|_{\partial \Omega_\varepsilon} = 0 \right\}$$

$$\downarrow$$

$$\tilde{\psi} = U\psi ; \quad \left\{ \pm \frac{\partial \tilde{\psi}}{\partial u} \bigg|_{u=\pm 1} + \left[ \alpha \mp \frac{\varepsilon^{a-1} \gamma(s/\varepsilon)}{2(1 \pm \varepsilon^{a-1} \gamma(s/\varepsilon))} \right] \tilde{\psi} \bigg|_{u=\pm 1} = 0 \right\} \quad (1)$$

- Normal Modes: $\phi_{\varepsilon, n}(s)$ satisfy the boundary conditions (1) for all $s \in \mathbb{R}$ and

$$-\frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} \phi_{\varepsilon, n}(s) = \frac{\lambda_{\varepsilon, n}(s; \alpha)}{\varepsilon^{2a}} \phi_{\varepsilon, n}(s) \quad \forall s \in \mathbb{R}$$
Step 1: Reduced Hamiltonian

- The operator $-\Delta_{\Omega_{\varepsilon}}^R$ on $L^2(\Omega_{\varepsilon})$ is unitarily equivalent to $H_\varepsilon$ on $L^2(\mathbb{R} \times (-1, 1))$

$$H_\varepsilon = -\frac{\partial}{\partial s} \frac{1}{(1 + \varepsilon^{a-1} u \gamma(s/\varepsilon))^2} \frac{\partial}{\partial s} - \frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} + W_\varepsilon(s, u),$$

- The Boundary Conditions are not preserved

$$\psi \in \mathcal{D}(-\Delta_{\Omega_{\varepsilon}}^R) ; \quad \left\{ \frac{\partial \psi}{\partial n} \right\}_{\partial \Omega_{\varepsilon}} + \alpha \psi |_{\partial \Omega_{\varepsilon}} = 0$$

$$\Downarrow$$

$$\tilde{\psi} = U\psi ; \quad \left\{ \pm \frac{\partial \tilde{\psi}}{\partial u} \bigg|_{u=\pm 1} + \left[ \alpha \mp \frac{\varepsilon^{a-1} \gamma(s/\varepsilon)}{2(1 \pm \varepsilon^{a-1} \gamma(s/\varepsilon))} \right] \tilde{\psi} \bigg|_{u=\pm 1} = 0 \right\} \quad (1)$$

- Normal Modes: $\phi_{\varepsilon,n}(s)$ satisfy the boundary conditions (1) for all $s \in \mathbb{R}$ and

$$-\frac{1}{\varepsilon^{2a}} \frac{\partial^2}{\partial u^2} \phi_{\varepsilon,n}(s) = \frac{\lambda_{\varepsilon,n}(s; \alpha)}{\varepsilon^{2a}} \phi_{\varepsilon,n}(s) \quad \forall s \in \mathbb{R}$$

- Reduced Hamiltonian

$$(\phi_{\varepsilon,m}(s), H_\varepsilon \phi_{\varepsilon,n}(s))_{L^2((-1,1))} \simeq \delta_{m,n} \left( -\frac{d^2}{ds^2} + \frac{\beta_n(\alpha)}{\varepsilon^2} \gamma^2(s/\varepsilon) + \frac{\lambda_n^{(0)}(\alpha)}{\varepsilon^{2a}} \right)$$
Step 1: $\beta_n(\alpha)$
Summary of the Proof

Step 1  Reduction to a One Dimensional Dynamics (dependent on $\varepsilon$)

Step 2  Analysis of the One Dimensional Problem (limit $\varepsilon \to 0$)
Definition (Zero energy resonance)

Assume that $e^{|c|}v \in L^1(\mathbb{R})$ for some $c > 0$. We say that the Hamiltonian

$$h = -\frac{d^2}{ds^2} + v(s)$$

has a zero energy resonance if there exists $\psi_r \in L^\infty(\mathbb{R})$, $\psi_r \notin L^2(\mathbb{R})$ such that $h\psi_r = 0$ in distributional sense.
**Definition (Zero energy resonance)**
Assume that $e^{c|\cdot|}v \in L^1(\mathbb{R})$ for some $c > 0$. We say that the Hamiltonian

$$h = -\frac{d^2}{ds^2} + v(s)$$

has a zero energy resonance if there exists $\psi_r \in L^\infty(\mathbb{R}), \psi_r \notin L^2(\mathbb{R})$ such that $h\psi_r = 0$ in distributional sense.

**Proposition**
Assume that $e^{c|\cdot|}v \in L^1(\mathbb{R})$ for some $c > 0$. Then two cases can occur:
Definition (Zero energy resonance)
Assume that \( |\cdot|^c v \in L^1(\mathbb{R}) \) for some \( c > 0 \). We say that the Hamiltonian
\[
h = -\frac{d^2}{ds^2} + v(s)
\]
has a zero energy resonance if there exists \( \psi_r \in L^\infty(\mathbb{R}) \), \( \psi_r \notin L^2(\mathbb{R}) \) such that \( h\psi_r = 0 \) in distributional sense.

Proposition
Assume that \( e^{c\cdot}|\cdot| v \in L^1(\mathbb{R}) \) for some \( c > 0 \). Then two cases can occur:

1. \( h \) has no zero energy resonances
Definition (Zero energy resonance)
Assume that $e^{c|\cdot|}v \in L^1(\mathbb{R})$ for some $c > 0$. We say that the Hamiltonian

$$h = -\frac{d^2}{ds^2} + v(s)$$

has a zero energy resonance if there exists $\psi_r \in L^\infty(\mathbb{R})$, $\psi_r \notin L^2(\mathbb{R})$ such that $h\psi_r = 0$ in distributional sense.

Proposition
Assume that $e^{c|\cdot|}v \in L^1(\mathbb{R})$ for some $c > 0$. Then two cases can occur:

1. $h$ has no zero energy resonances
2. $h$ has a zero energy resonance. In such a case one can define

$$\rho_1 = \lim_{x \to -\infty} \psi_r(x) \quad \text{and} \quad \rho_2 = \lim_{x \to +\infty} \psi_r(x)$$

Function $\psi_r$ is unique up to a multiplicative constant and can be chosen in such a way that constants $\rho_1$ and $\rho_2$ are real, and $\rho_1^2 + \rho_2^2 = 1$.
Lemma

Let us denote by $h_\varepsilon$ the one dimensional Hamiltonian

$$h_\varepsilon = -\frac{d^2}{ds^2} + \frac{1}{\varepsilon^2} v(\cdot/\varepsilon)$$

Assume that $e^{c|\cdot|} v \in L^1(\mathbb{R})$ for some $c > 0$. Then two cases can occur:
Step 2: One Dimensional Scaled Hamiltonians

Lemma

Let us denote by $h_\varepsilon$ the one dimensional Hamiltonian

$$h_\varepsilon = -\frac{d^2}{ds^2} + \frac{1}{\varepsilon^2} v(\cdot/\varepsilon)$$

Assume that $e^{c|\cdot|} v \in L^1(\mathbb{R})$ for some $c > 0$. Then two cases can occur:

1. There does not exist a zero energy resonance for $h = -\frac{d^2}{ds^2} + v(s)$

   $$u - \lim_{\varepsilon \to 0} (h_\varepsilon - k^2)^{-1} = (-\Delta^\text{dec}_s - k^2)^{-1}$$

   $$k^2 \in \mathbb{C}\setminus\mathbb{R}, \text{Im} \ k > 0$$

   $$-\Delta^\text{dec}_s f = -\frac{d^2 f}{ds^2} + f(0^-) = f(0^+) = 0$$
Lemma

Let us denote by \( h_\varepsilon \) the one dimensional Hamiltonian

\[
h_\varepsilon = -\frac{d^2}{ds^2} + \frac{1}{\varepsilon^2} v(\cdot / \varepsilon)
\]

Assume that \( e^{c|\cdot|} v \in L^1(\mathbb{R}) \) for some \( c > 0 \). Then two cases can occur:

1. There does not exist a zero energy resonance for \( h = -\frac{d^2}{ds^2} + v(s) \)

\[
\lim_{\varepsilon \to 0} (h_\varepsilon - k^2)^{-1} = (-\Delta_s^{\text{dec}} - k^2)^{-1} \quad k^2 \in \mathbb{C} \setminus \mathbb{R}, \quad \text{Im } k > 0
\]

\[
-\Delta_s^{\text{dec}} f = -\frac{d^2 f}{ds^2} + f(0^-) = f(0^+) = 0
\]

2. There exists a zero energy resonance for \( h = -\frac{d^2}{ds^2} + v(s) \)

\[
\lim_{\varepsilon \to 0} (h_\varepsilon - k^2)^{-1} = (-\Delta_s^{\rho_1\rho_2} - k^2)^{-1} \quad k^2 \in \mathbb{C} \setminus \mathbb{R}, \quad \text{Im } k > 0
\]

\[
-\Delta_s^{\rho_1\rho_2} f = -\frac{d^2 f}{ds^2} + \rho_2 f(0^-) = \rho_1 f(0^+) \quad \rho_1 f'(0^-) - \rho_2 f'(0^+) = 0
\]
Theorem

Assume that $C$ has no self intersections and that $\gamma \in C^\infty_0(\mathbb{R})$. Then for all $a > 3$ and for all $n = 0, 1, 2, \ldots$, two cases can occur:

1. There does not exist a zero energy resonance for $h_{n = -d^2 s^2 + \beta_n(\alpha)}^n(\gamma) := \phi_{\varepsilon, m}^{-1} \left( H_\varepsilon - k^2 - \frac{\lambda_n^{(0)}(\alpha)}{\varepsilon^{2a}} \right) \phi_{\varepsilon, n}^{-1}$ $k^2 \in \mathbb{C}\backslash \mathbb{R}, \ \text{Im} \ k > 0$

2. There exists a zero energy resonance for $h_{n = -d^2 s^2 + \beta_n(\alpha)}^n(\gamma) := \phi_{\varepsilon, m}^{-1} \left( H_\varepsilon - k^2 - \frac{\lambda_n^{(0)}(\alpha)}{\varepsilon^{2a}} \right) \phi_{\varepsilon, n}^{-1}$ $k^2 \in \mathbb{C}\backslash \mathbb{R}, \ \text{Im} \ k > 0$

\[ \rho_n, 1 = \lim_{x \to -\infty} \psi_n, r(x), \quad \rho_n, 2 = \lim_{x \to +\infty} \psi_n, r(x) \]
Main Result

- Reduced resolvent: \( \overline{R}_{m,n}(k^2) \in \mathcal{B}(L^2(\mathbb{R}), L^2(\mathbb{R})) \)

\[
\overline{R}_{m,n}(\varepsilon; k^2) := \left( \phi_{\varepsilon,m}, \left( H_{\varepsilon} - k^2 - \frac{\lambda_n^{(0)}(\alpha)}{\varepsilon^{2a}} \right)^{-1} \phi_{\varepsilon,n} \right)_{L^2((-1,1))} \quad k^2 \in \mathbb{C} \setminus \mathbb{R}, \ \text{Im} \ k > 0
\]

Theorem

Assume that \( C \) has no self intersections and that \( \gamma \in C_0^\infty(\mathbb{R}) \). Then for all \( a > 3 \) and for all \( n = 0, 1, 2, \ldots \), two cases can occur:
Main Result

- Reduced resolvent: $\overline{R}_{m,n}(k^2) \in \mathcal{B}(L^2(\mathbb{R}), L^2(\mathbb{R}))$

$$\overline{R}_{m,n}(\varepsilon; k^2) := \left(\phi_{\varepsilon,m}, \left(H_{\varepsilon} - k^2 - \frac{\lambda_n^{(0)}(\alpha)}{\varepsilon^{2a}}\right)^{-1} \phi_{\varepsilon,n}\right)_{L^2((-1,1))}$$

k$^2 \in \mathbb{C}\setminus\mathbb{R}$, Im $k > 0$

Theorem

Assume that $C$ has no self intersections and that $\gamma \in C_0^\infty(\mathbb{R})$. Then for all $a > 3$ and for all $n = 0, 1, 2, \ldots$, two cases can occur:

1. There does not exist a zero energy resonance for $h_n = -\frac{d^2}{ds^2} + \beta_n(\alpha)\gamma^2$

$$u - \lim_{\varepsilon \to 0} \overline{R}_{m,n}^\varepsilon(k^2) = \delta_{m,n}(\Delta_{s}^{\text{dec}} - k^2)^{-1} \quad k^2 \in \mathbb{C}\setminus\mathbb{R}$, Im $k > 0$$
Main Result

- Reduced resolvent: \( \overline{R}_{m,n}(k^2) \in \mathcal{B}(L^2(\mathbb{R}), L^2(\mathbb{R})) \)

\[
\overline{R}_{m,n}(\varepsilon; k^2) := \left( \phi_{\varepsilon,m}, \left( H_{\varepsilon} - k^2 - \frac{\lambda_n^{(0)}(\alpha)}{\varepsilon^{2a}} \right)^{-1} \phi_{\varepsilon,n} \right)_{L^2((-1,1))} \quad k^2 \in \mathbb{C}\setminus\mathbb{R}, \Im k > 0
\]

Theorem

Assume that \( C \) has no self intersections and that \( \gamma \in C^\infty_0(\mathbb{R}) \). Then for all \( a > 3 \) and for all \( n = 0, 1, 2, \ldots \), two cases can occur:

1. **There does not exist a zero energy resonance** for \( h_n = -\frac{d^2}{ds^2} + \beta_n(\alpha)\gamma^2 \)

   \[
u - \lim_{\varepsilon \to 0} \overline{R}_{m,n}(k^2) = \delta_{m,n}(-\Delta^\text{dec}_s - k^2)^{-1} \quad k^2 \in \mathbb{C}\setminus\mathbb{R}, \Im k > 0
\]

2. **There exists a zero energy resonance** for \( h_n = -\frac{d^2}{ds^2} + \beta_n(\alpha)\gamma^2 \)

   \[
u - \lim_{\varepsilon \to 0} \overline{R}_{m,n}(k^2) = \delta_{m,n}(-\Delta^\rho_{s,1}\rho_{n,2} - k^2)^{-1} \quad k^2 \in \mathbb{C}\setminus\mathbb{R}, \Im k > 0
\]

\[
\rho_{n,1} = \lim_{x \to -\infty} \psi_{n,r}(x) \quad \text{and} \quad \rho_{n,2} = \lim_{x \to +\infty} \psi_{n,r}(x)
\]
Interplay between Curvature and Boundary Condition

\[ h_n := -\frac{d^2}{ds^2} + \beta_n(\alpha)\gamma^2(s) \]
Conclusions

Interplay between Curvature and Boundary Condition

\[ h_n := -\frac{d^2}{ds^2} + \beta_n(\alpha) \gamma^2(s) \]
Conclusions

- Dirichlet case [Albeverio, Cacciapuoti, Finco ’07]: \( \alpha = \infty, \beta_n(\alpha) = -1/4 \)
  - No dependence on the transverse energy
Conclusions

- Dirichlet case [Albeverio, Cacciapuoti, Finco ’07]: \( \alpha = \infty, \beta_n(\alpha) = -1/4 \)
  - No dependence on the transverse energy

- Neumann case [Cacciapuoti, Finco ’08]: \( \alpha = 0 \)

- Robin case [Cacciapuoti, Finco ’08]:
  - Dependence on the transverse energy
  - Existence of a generic and non-generic case
Conclusions

- Dirichlet case [Albeverio, Cacciapuoti, Finco ’07]: $\alpha = \infty$, $\beta_n(\alpha) = -1/4$
  - No dependence on the transverse energy

- Neumann case [Cacciapuoti, Finco ’08]: $\alpha = 0$
  - Reduction of the dynamics with respect to the ground transverse state: $\beta_0(0) = 0$, standard condition in the vertex
  \[
  (\phi_0^N, H_\epsilon^N \phi_0^N)_{L^2((-1,1))} \sim -\frac{d^2}{ds^2}
  \]
Conclusions

- Dirichlet case [Albeverio, Cacciapuoti, Finco '07]: $\alpha = \infty$, $\beta_n(\alpha) = -1/4$
  - No dependence on the transverse energy

- Neumann case [Cacciapuoti, Finco '08]: $\alpha = 0$
  - Reduction of the dynamics with respect to the ground transverse state: $\beta_0(0) = 0$, standard condition in the vertex
    \[
    (\phi_0^N, H^N_\epsilon \phi_0^N)_{L^2((-1,1))} \approx -\frac{d^2}{ds^2}
    \]
  - Reduction of the dynamics with respect to excited transverse states, $n > 0$: $\beta_n(0) = 3/4$, decoupling conditions in the vertex (no possibility of zero energy resonances!)
    \[
    (\phi_n^N, H^N_\epsilon \phi_n^N)_{L^2((-1,1))} \approx \left(-\frac{d^2}{ds^2} + \frac{3}{4} \frac{1}{\epsilon^2} \gamma^2(s/\epsilon) + \frac{E^N_n}{\epsilon^{2a}}\right) \quad n \geq 1
    \]

- Robin case [Cacciapuoti, Finco '08]:
  - Dependence on the transverse energy
  - Existence of a generic and non-generic case
Conclusions

- Dirichlet case [Albeverio, Cacciapuoti, Finco ’07]: $\alpha = \infty$, $\beta_n(\alpha) = -1/4$
  - No dependence on the transverse energy

- Neumann case [Cacciapuoti, Finco ’08]: $\alpha = 0$
  - Reduction of the dynamics with respect to the ground transverse state: $\beta_0(0) = 0$, standard condition in the vertex
    $$\left( \phi_0^N, H_\varepsilon \phi_0^N \right)_{L^2((-1,1))} \simeq \frac{-d^2}{ds^2}$$
  - Reduction of the dynamics with respect to excited transverse states, $n > 0$: $\beta_n(0) = 3/4$, decoupling conditions in the vertex (no possibility of zero energy resonances!)
    $$\left( \phi_n^N, H_\varepsilon \phi_n^N \right)_{L^2((-1,1))} \simeq \left( -\frac{d^2}{ds^2} + \frac{3}{4} \frac{1}{\varepsilon^2} \gamma^2(s/\varepsilon) + \frac{E_n^N}{\varepsilon^{2a}} \right) \quad n \geq 1$$

- Robin case [Cacciapuoti, Finco ’08]:
  - Dependence on the transverse energy
  - Existence of a generic and non-generic case
Small perturbations of the geometry [Cacciapuoti, Exner '07]
Small perturbations of the geometry [Cacciapuoti, Exner '07]

\[ \gamma_\varepsilon(s) := \frac{\sqrt{1 + \lambda \varepsilon}}{\varepsilon} \gamma\left(\frac{s}{\varepsilon}\right) \quad \lambda \in \mathbb{R} \]

\[ \theta_\varepsilon = \left(1 + \frac{\lambda}{2\varepsilon}\right)\theta + \mathcal{O}(\varepsilon^2) \]
Small perturbations of the geometry [Cacciapuoti, Exner '07]

\[ \gamma_{\varepsilon}(s) := \frac{\sqrt{1 + \lambda \varepsilon}}{\varepsilon} \gamma\left(\frac{s}{\varepsilon}\right) \quad \lambda \in \mathbb{R} \]

\[ \theta_{\varepsilon} = \left(1 + \frac{\lambda}{2}\varepsilon\right) \theta + \mathcal{O}(\varepsilon^2) \]

In the non generic case the parameter \( \lambda \) enters into the definition of the boundary conditions of the operator on the graph

\[ \rho_2 f_1(0) = \rho_1 f_2(0) \]

\[ \rho_1 f_1'(0) + \rho_2 f_2'(0) = \frac{\hat{\lambda}}{2} \left( \rho_1 f_1(0) + \rho_2 f_2(0) \right) \]

\[ \hat{\lambda} := -\lambda \int_{-\infty}^{\infty} \frac{\gamma(s)^2}{4} |\psi'(s)| ds \]
Conclusions

Graphs with more than two edges

- **Neumann case**: Rubinstein and Schatzman '01, Kuchment and Zeng '01, Post '06
  - Lowest transverse mode, resolvent convergence to the standard condition in the vertex

- **Dirichlet case**: Post '05
  - Compact graphs, assumption of "small vertices"
  - Lowest transverse mode, spectral convergence to decoupling conditions

- **Generic boundary conditions on \( \partial \Omega \)**: Grieser '08
  - Compact graphs
  - Lowest and excited transverse modes, convergence of the spectrum

- **Non compact graphs.**
  - Lowest transverse mode, convergence of scattering states
Conclusions

Graphs with more than two edges

- **Neumann case**: Rubinstein and Schatzman ’01, Kuchment and Zeng ’01, Post ’06
  - Lowest transverse mode, resolvent convergence to the standard condition in the vertex

- **Dirichlet case**: Post ’05
  - Compact graphs, assumption of “small vertices”
  - Lowest transverse mode, spectral convergence to decoupling conditions
Graphs with more than two edges

- **Neumann case:** Rubinstein and Schatzman ’01, Kuchment and Zeng ’01, Post ’06
  - Lowest transverse mode, resolvent convergence to the standard condition in the vertex

- **Dirichlet case:** Post ’05
  - Compact graphs, assumption of “small vertices”
  - Lowest transverse mode, spectral convergence to decoupling conditions

- **Generic boundary conditions on $\partial \Omega$:** Grieser ’08
  - Compact graphs
  - Lowest and excited transverse modes, convergence of the spectrum
Conclusions

Graphs with more than two edges

- Neumann case: Rubinstein and Schatzman '01, Kuchment and Zeng '01, Post '06
  - Lowest transverse mode, resolvent convergence to the standard condition in the vertex

- Dirichlet case: Post '05
  - Compact graphs, assumption of “small vertices”
  - Lowest transverse mode, spectral convergence to decoupling conditions

- Generic boundary conditions on $\partial \Omega$: Grieser '08
  - Compact graphs
  - Lowest and excited transverse modes, convergence of the spectrum

- Dirichlet case: Costa and Dell’Antonio '09
  - Non compact graphs.
  - Lowest transverse mode, convergence of scattering states