A SINGULAR DIFFERENTIAL OPERATOR: TITCHMARSH-WEYL COEFFICIENTS AND OPERATOR MODELS

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Abstract. In this note the connection between a (generalized) Titchmarsh-Weyl-coefficient for a singular Sturm-Liouville-operator and a certain singular perturbation of this operator is established.

1. Introduction

The singular differential expression
\[ \ell(y) := -y''(x) + \left( \frac{q_0}{x^2} + \frac{q_1}{x} \right) y(x) \quad \text{on } x \in (0, \infty) \]
is a rather well studied object, due to the fact that it appears as radial part when separating variables for the Schrödinger operator with Coulomb potential in 2 or 3 dimensions. We are comparing here two - at first sight - rather different approaches, and show that they are actually connected in a similar way as in the classical case.

As a motivation let us briefly recall the situation in the case of a Sturm-Liouville-operator \( \ell(y) := -y'' + qy \) on the half line \([0, \infty)\), which is regular at 0, that is for the real potential \( q \) it holds \( q \in L_{loc}^1[0, \infty) \), and which is in limit point case at \( \infty \) (in the sense of H. Weyl). Under these assumptions for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the equation
\[ \ell(y) = \lambda y \]
has exactly only one (up to scalar multiples) solution which belongs to \( L^2(0, \infty) \). Hence with the basic solutions \( y_1 \) and \( y_2 \) of (1.2), which are determined by the Cauchy data
\[ y_1(0, \lambda) = 0 \quad y_2(0, \lambda) = 1 \]
\[ y'_1(0, \lambda) = -1 \quad y'_2(0, \lambda) = 0, \]
the requirement
\[ g(x, \lambda) := y_2(x, \lambda) - m(\lambda)y_1(x, \lambda) \in L^2(0, \infty) \]
defines \( m(\lambda) \) uniquely. This function is usually called Titchmarsh-Weyl coefficient of the differential expression \( \ell \). It is a Nevanlinna function, \( m \in \mathcal{N}_0 \), that is, it maps the upper half plane \( \mathbb{C}^+ \) holomorphically into itself. Its analytic properties are closely connected with the spectrum of the self adjoint realizations of \( \ell \). These realizations, or in other words, self adjoint extensions of the corresponding minimal
operator, are given as restrictions $L_\tau$, $\tau \in \mathbb{R} \cup \{\infty\}$, of the differential expression $\ell$ to the domain

$$\text{dom}(L_\tau) = \{y \in L^2(0,\infty), \ell(y) \in L^2(0,\infty), y(0) - \tau y'(0) = 0\}.$$ 

They are connected via

$$(L_\tau - \lambda)^{-1} = (L_0 - \lambda)^{-1} - \frac{\langle g(x, \lambda), \cdot \rangle}{m(\lambda) - \frac{1}{\tau}} g(x, \lambda).$$

Note that here and in the following we use the notation $\langle \cdot, \cdot \rangle$ for the inner product, such that it is linear in the second and conjugate linear in the first argument.

On the other side the same differential expression can be considered using methods of perturbation theory. Define the element $\varphi := (L_0 - \lambda_0)g(\cdot, \lambda_0)$, which in general does not belong to $L^2(0,\infty)$, but rather $\varphi \in \mathcal{H}_{-2}(L_0)$ since $g \in L^2(0,\infty)$.

For more details on the rigged spaces $\mathcal{H}_{-n}$ see Section 2.2 below, cf. also [1]. Then by standard techniques with the singular perturbation

$$(L_\gamma - \lambda)^{-1} = (L_0 - \lambda)^{-1} - \frac{Q(\lambda) - m(\lambda)}{Q(\lambda) + \gamma} g(\cdot, \lambda)$$

there is associated a whole family of self adjoint operators in $L^2(0,\infty)$, which are given by

$$(L_\gamma - \lambda)^{-1} = (L_0 - \lambda)^{-1} - \frac{\langle (L_0 - \lambda)^{-1} \varphi, \cdot \rangle}{Q(\lambda) + \gamma}(L_0 - \lambda)^{-1} \varphi, \quad \gamma \in \mathbb{R} \cup \{\infty\}.$$ 

Here $Q$ is a $Q$-function corresponding to the symmetry $S$, which is defined as the restriction of $L_0$ to those elements $y$ for which $\langle \varphi, y \rangle = 0$ and its self adjoint extension $L_0$. Since $g(\cdot, \lambda) = (L_0 - \lambda)\varphi$ the formulas (1.6) and (1.4) describe the same family of self adjoint extensions. Moreover, $Q(\lambda) - m(\lambda)$ is a real constant.

It is the aim of this paper to lift this picture to a more general situation. We consider the singular differential expression

$$\ell(y) := -y''(x) + \frac{q_0 + qx}{x^2} y(x) \quad \text{for } x \in (0,\infty)$$

with $q_1 \in \mathbb{R}$ and $q_0 > -\frac{1}{3}$. These assumptions guarantee that $\ell$ is in limit point case at $\infty$ and the corresponding operators are semi-bounded from below.

In [5] a generalized Titchmarsh-Weyl-coefficient has been introduced like in (1.3), however the normalization of the solutions $y_1$ and $y_2$ has to be done differently, see Proposition 2.1. In this situation $m$ need not be a Nevanlinna function anymore, but it was shown in [6] that it belongs to some generalized Nevanlinna class $\mathcal{N}_\kappa$ and an estimate for $\kappa$, the index of non-positivity, was given. See also [7], where generalized Titchmarsh-Weyl coefficients are studied also for more general differential expressions.

On the other side, in the particular case $q_1 = 0$, that is, when the differential operator is the so-called Bessel operator, a certain (super) singular perturbation of the differential operator - in a transformed version - has been studied in [4]. Here for large $q_0$ the element $\varphi$ will be very singular, $\varphi \in \mathcal{H}_{-n}$ with $n > 2$. Then for the interpretation of (1.5) a finite dimensional extension of the original space is needed. In this construction a generalized Nevanlinna function $Q$ plays a crucial role. In the case of the Bessel operator it turned out that this function $Q$ and the Titchmarsh-Weyl-coefficient $m$ in Fulton’s approach coincide up to a polynomial of certain degree.
It is the aim of this paper to explore the connection between these two approaches in the general case. In Section 2, following [5], the proper generalizations of \( y_1 \) and \( y_2 \) are studied in detail, where the emphasis is put on asymptotic expansions for the singular solution and its regularizations. Then we change to an operator theoretic point of view. For \(-\frac{1}{4} < q_0 < \frac{3}{4}\) at the left endpoint \( x = 0 \) limit circle case prevails, hence this case is rather close to the regular situation and is briefly recalled in Subsection 2.2. For the limit point case \((q_0 \geq \frac{3}{4})\) the above mentioned expansions are used in Subsection 2.3 in order to show that \( \varphi \) belongs to some \( \mathcal{H}_{-n}(L_0) \) where \( n \) is determined explicitly. Section 3 contains a short description of the model for a (super) singular perturbation according to [8], which then is directly applied to (1.5) in the limit point case. In fact, here we are using a different model than [4]. However there is a very close connection, see [3], in particular, the same generalized Nevanlinna function \( Q \in \mathcal{N}_\kappa \) appears.

The main result in this paper gives then the direct link between the main objects in the above approaches. We show in Theorem 3.1 that the Titchmarsh-Weyl-coefficient studied by [5] (cf. also (2.7)) and the generalized Nevanlinna function \( Q \), which plays the role of a Q-function in the model for the singular perturbation actually differ only by a polynomial of low degree. This also reproves the result by [6] and even gives the precise number for \( \kappa \).

2. Generalized Titchmarsh-Weyl-coefficient and regularized solutions

2.1. Asymptotics at the singularity. In this section we study the solutions of the differential equation

\[ -y''(x) + \left( \frac{q_0}{x^2} + \frac{q_1}{x} \right) y(x) = \lambda y(x) \quad x \in (0, \infty), \lambda \in \mathbb{C}, \]

where \( q_1 \in \mathbb{R} \) and \( q_0 > -\frac{1}{4} \) with respect to their asymptotic behaviour at the singularity \( x = 0 \). We follow the lines of [5] and will also extend the analysis there.

Note that the point \( x = 0 \) is a so-called weak singularity of equation (2.1), thus one makes the generalized power series Ansatz

\[ y(x, \lambda) = x^\alpha \sum_{j=0}^{\infty} a_j(\lambda) x^j \quad \text{with } a_0 \neq 0. \]

The corresponding index equation turns out to be

\[ \alpha^2 - \alpha - q_0 = 0 \]

and it has the real solutions \( \alpha_{\pm} := \frac{1}{2} \pm \sqrt{\frac{1}{4} + q_0} \), with \( \alpha_- < \alpha_+ \) and \( \alpha_- + \alpha_+ = 1 \).

Furthermore, for the coefficients one obtains the recursion

\[ a_1 = \frac{q_1}{2\alpha} a_0 \quad \text{and} \quad a_{j+2} = \frac{q_1 a_{j+1} - \lambda a_j}{(j+2)(2\alpha + j + 1)}, \quad j = 0, 1, \ldots. \]

There are two particular solutions \( g_+(x, \lambda) \) and \( g_-(x, \lambda) \) (the so-called “regular” and “singular” solution) corresponding to the indices \( \alpha_+ \) and \( \alpha_- \), correspondingly. The following lemma summarizes some of their asymptotic properties.
Proposition 2.1. With $\alpha_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + q_0}$ equation (2.1) has two linearly independent solutions

\begin{equation}
    g_+(x, \lambda) = \sum_{j=0}^{\infty} a_j(\lambda)x^{\alpha_+ + j}
\end{equation}

(2.3)

and

\begin{equation}
    g_-(x, \lambda) = \sum_{j=0}^{m_0 - 1} c_j(\lambda)x^{\alpha_- + j} + o(x^{\alpha_- + 1}) \quad \text{for } x \to +
\end{equation}

with $m_0 := [\alpha_+ - \alpha_-]$ and coefficients $a_j$ and $c_j$ satisfying the recursion

\begin{equation}
    x_{j+2} = \frac{q_1 x_{j+1} - \lambda x_j}{(j + 2)(2\alpha + j + 1)}
\end{equation}

with $\alpha = \alpha_+$ and starting values $a_0 = 1$, $a_1 = \frac{\alpha}{2\alpha_+}$ and with $\alpha = \alpha_-$ and starting values $c_0 = \frac{1}{\alpha_+ - \alpha_+}$, $c_1 = \frac{q_1}{2\alpha_-}c_0$, respectively. Moreover the following holds:

(i) The functions $g_+(\cdot, \lambda)$ and $g_-(\cdot, \lambda)$ and their derivatives with respect to $x$ are entire in $\lambda$ for all $x \in (0, \infty)$ and

\[ g_\pm(\cdot, \lambda) = g_\pm(\cdot, \bar{\lambda}) \quad \text{and} \quad g'_\pm(\cdot, \lambda) = g'_\pm(\cdot, \bar{\lambda}). \]

(ii) With the notation $W$ for the Wronskian

\[ W(g_1(x), g_2(x)) := g_1(x)g'_2(x) - g'_1(x)g_2(x). \]

it holds:

\begin{align*}
    \lim_{x \to +} W(g_+(\cdot, \lambda), g_-(\cdot, \lambda)) &= 1 \quad \forall \lambda \in \mathbb{C} \\
    \lim_{x \to 0} W(g_+(\cdot, \lambda), g_-(\cdot, z)) &= 1 \quad \forall \lambda, z \in \mathbb{C} \\
    \lim_{x \to 0} W(g_+(\cdot, \lambda), g_+(\cdot, z)) &= 0 \quad \forall \lambda, z \in \mathbb{C} \\
    \lim_{x \to 0} W(g_-(\cdot, \lambda), g_-(\cdot, z)) &= \begin{cases} \infty & \text{if } q_0 < \frac{3}{4} \\ 0 & \text{if } q_0 \geq \frac{3}{4} \end{cases} \quad \forall \lambda, z \in \mathbb{C}.
\end{align*}

Proof. From (2.2) the expansion of $g_+$ follows directly. The classical theory shows that in order to obtain a second linear independent solution one has to distinguish two different cases. If $\alpha_+ - \alpha_- \notin \mathbb{N}$, then a “singular” solution is of the form

\begin{equation}
    g_-(x, \lambda) := \sum_{j=0}^{\infty} c_j(\lambda)x^{\alpha_- + j} \quad \text{with } c_0 = \frac{1}{\alpha_+ - \alpha_+},
\end{equation}

where $c_j$ satisfy the recursion (2.2) with $\alpha = \alpha_-$. Here the choice of $c_0$ is done such that the Wronskian in (ii) is normalized. If $\alpha_+ - \alpha_- = m_0 \in \mathbb{N} \setminus \{0\}$ then the second solution can be obtained by the Ansatz

\begin{equation}
    g_-(x, \lambda) = K(\lambda) \ln x g_+(x, \lambda) + \sum_{j=0}^{\infty} d_j(\lambda)x^{\alpha_- + j}.
\end{equation}

This yields

\[
    d_0 = \frac{1}{\alpha_+ - \alpha_+}, \quad d_1 = \frac{q_1}{2\alpha_+}d_0, \quad d_{j+2} = \frac{q_1 d_{j+1} - \lambda d_j}{(j + 2)(2\alpha_+ + j + 1)}, \quad 0 \leq j < m_0 - 2,
\]

and

\[
    d_{j+2} = -\frac{q_1 d_{j+1} - \lambda d_j}{(j + 2)(m_0 - 2 - j)} + K(\lambda) a_{j+2-m_0} \frac{2j + 4 - m_0}{(j + 2)(m_0 - 2 - j)} \quad k > m_0 - 2
\]
with
\[ K(\lambda) = \frac{d_{m_0-1}q_1 - d_{m_0-2}}{m_0}. \]
Summed up, this shows that in both cases the claimed expansion for \( g_- \) holds. The other statements follow then directly from the asymptotic expansions (2.3). For more details see also [5].

Note that in the above proof in the special case \( \alpha_+ - \alpha_- \in \mathbb{N} \) there was no requirement on the coefficient \( d_{m_0} \) since then \( \alpha_- + m_0 = \alpha_+ \). Choosing \( d_{m_0} \) appropriately gives a further refinement of expansion (2.3) for \( g_- \).

**Corollary 2.1.** Equation (2.1) has a solution \( g_- \) which is of the form
\[ g_-(x, \lambda) = \sum_{j=0}^{m_0} c_j(\lambda)x^{\alpha_- + j} + K(\lambda)x^{\alpha_+} \ln x + x^{m_0 + \alpha_- + 1}(H_1(x) + \ln x H_2(x)), \]
where \( m_0 = \lfloor \alpha_+ - \alpha_- \rfloor \), the coefficients \( c_j(\lambda) \) and \( K(\lambda) \) are polynomials in \( \lambda \) of degree \( \leq \frac{m_0}{2} \), and the functions \( H_1 \) and \( H_2 \) are holomorphic at \( x = 0 \).

**Proof.** If \( \alpha_+ - \alpha_- \notin \mathbb{N} \) then (2.4) gives directly \( K(\lambda) = 0 \) and \( H_2(x) = 0 \) and, indeed recursion (2.2) implies that \( c_j \) is a polynomial in \( \lambda \) of degree \( \lfloor \frac{j}{2} \rfloor \). If \( \alpha_+ - \alpha_- \in \mathbb{N} \), then (2.6) follows immediately from (2.5), and here only \( d_{m_0}(\lambda) \) has to be chosen as a polynomial in \( \lambda \) with degree \( \leq \frac{m_0}{2} \).

From Proposition (2.3) one sees that for some \( x_0 \in \mathbb{R} \) and for all \( \lambda \in \mathbb{C} \) it holds
\[ g_+(\cdot, \lambda) \in L^2(0, x_0) \text{ for all } q_0 > -\frac{1}{4}, \]
\[ g_-(\cdot, \lambda) \in L^2(0, x_0) \text{ if and only if } -\frac{1}{4} < q_0 < \frac{3}{4}, \]
that is, for the differential expression \( \ell \) prevails limit point case at the singular endpoint 0 if and only if \( q_0 \geq \frac{3}{4} \).

The endpoint \( \infty \), however, under our assumptions always is in limit point case. Thus there is one linear combination of \( g_+ \) and \( g_- \) which is square integrable in a neighbourhood of \( \infty \):
\[ g(\cdot, \lambda) := g_-(\cdot, \lambda) - m(\lambda)g_+(\cdot, \lambda) \in L^2(x_0, \infty) \text{ for some } x_0 \in \mathbb{R}. \]

The function \( m_0 \) which is defined by (2.7), has been investigated in [5] and [6].

Obviously, locally at 0 the function \( g \) behaves as the singular solution \( g_- \). However, the first two coefficients \( c_0 \) and \( c_1 \) in the expansion of \( g_- \) actually do not depend on the spectral parameter \( \lambda \). Thus for \( \mu_1 \neq \mu_2 \) expansions (2.3) imply that the difference \( g(x, \mu_1) - g(x, \mu_2) = o(x^{\alpha_- + 2}) \) for \( x \to 0+ \), and hence is more regular at the origin. This gives rise to the following definition.

Let \( \mu_1, \ldots, \mu_k \in \mathbb{R}^+ \) be mutually different for \( k \geq 1 \), then we define
\[ g_k(x) := \sum_{i=1}^{k} A_i^{(k)} g(x, \mu_i) \]
with \( A_i^{(1)} = 1 \) and for \( k > 1 \)
\[ A_i^{(k)} = -\frac{1}{\mu_k - \mu_i} A_i^{(k-1)} \text{ for } i = 1, \ldots, k-1 \text{ and } A_k^{(k)} = -\sum_{i=1}^{k-1} A_i^{(k)}. \]

**Remark 2.2.** Note that it holds \( (\ell - \mu_k)g_k = g_{k-1} \) for \( k > 1 \), and \( (\ell - \mu_1)g_1 = 0 \), where \( \ell \) denotes the differential expression (2.1).
The next lemma shows that the regularity of these functions indeed increases with \( k \). In Theorem 2.7 we will also give an operator theoretic interpretation. Let us first introduce the number

\[
2(n := 2 + \left\lfloor \sqrt{\frac{1}{4} + q_0} \right\rfloor,
\]

which will turn out to be intimately connected with the problem under consideration. Here \([x]\) denotes the integer part of \( x \).

**Lemma 2.3.** Assume \( q_0 > \frac{3}{4} \), let \( k \leq n - 2 = \left\lfloor \sqrt{\frac{1}{4} + q_0} \right\rfloor \), and \( m_0 = [\alpha_+ - \alpha_-] \). Then the functions \( g_k(x) \) (defined in (2.8)) have the asymptotic expansions

\[
g_k(x) = \sum_{j=2(k-1)}^{m_0-1} C_j^{(k)} x^{\alpha_- + j} + o(x^{\alpha_-+1}) \quad \text{as } x \to 0
\]

where the first coefficient \( C_{2(k-1)}^{(k)} \neq 0 \).

**Proof.** Note first that under the above assumptions the sum in (2.11) is not empty. From (2.3) it follows

\[
g_k(x) = \sum_{j=0}^{m_0-1} C_j^{(k)} x^{\alpha_- + j} + o(x^{\alpha_-+1}) \quad \text{as } x \to 0+
\]

with \( C_j^{(k)} := \sum_{i=1}^k A_i^{(k)} c_j(\mu_i) \), where the coefficients \( A_i^{(k)} \) are given in (2.9) and \( c_j \) as in Proposition 2.1. We have to show that \( C_j^{(k)} = 0 \) for \( j < 2(k-1) \). For \( k = 1 \) the above statement is already included in Proposition 2.1. Hence let us now assume \( k > 1 \). Since \( c_0(\lambda) \) and \( c_1(\lambda) \) do not depend on \( \lambda \) we have directly

\[
C_j^{(k)} = c_j \sum_{i=1}^k A_i^{(k)} = 0 \quad \text{for } j = 0, 1 \text{ and } k > 1.
\]

For \( k > 1 \) the defining recursion (2.9) for the \( A_i^{(k)} \) gives

\[
C_j^{(k)} = \sum_{i=1}^k \frac{A_i^{(k-1)}}{\mu_k - \mu_i} (c_j(\mu_k) - c_j(\mu_i)).
\]

By using the recursion (2.2) for the \( c_j \) for \( j > 1 \) and then again (2.13) we get

\[
C_j^{(k)} = \frac{1}{j(2\alpha_- + j - 1)} (q_1 C_{j-1}^{(k)} - \mu_k C_{j-2}^{(k)} - C_{j-2}^{(k-1)}).
\]

Going on like this one obtains in finitely many steps that \( C_j^{(k)} \) is a linear combination of \( C_i^{(i)} \) and \( C_{j-1}^{(i)} \) for \( i \leq k \). Here \( 1 < i \leq k \) as long as \( j < 2(k-1) \) and hence \( C_j^{(k)} \) equals zero by (2.12). However, for \( j = 2(k-1) \) this implies furthermore \( C_{2(k-1)}^{(k)} = C \cdot c_0 \neq 0 \) with some nonzero constant \( C \).

**Remark 2.4.** For the first non-vanishing coefficients this implies the following recursion relation

\[
C_{2(k-1)}^{(k)} = -\frac{C_{2(k-2)}^{(k-1)}}{2(k-1)(\alpha_- + 2k - 3)}.
\]
2.2. Limit circle case: classical theory. Within this section we assume \( q_0 < \frac{1}{4} \).

In this case, as mentioned below, Proposition 2.1 implies that the expression \( \ell \) is in limit circle case also at the endpoint 0. We want to recall briefly this "regular" case in order to establish a connection to the following considerations in the "singular" situation.

With the differential expression (1.7)
\[
\ell(y)(x) := -y''(x) + \frac{q_0 + q_1x}{x^2} y(x) \quad \text{on } x \in (0, \infty)
\]
there is associated the maximal operator \( L_{\text{max}} \) by
\[
\text{dom } L_{\text{max}} := \{ y \in L^2(0, \infty) | y, y' \in AC_{\text{loc}}(0, \infty), \ell(y) \in L^2(0, \infty) \}
\]
and
\[
(L_{\text{max}}y)(x) := \ell(y)(x).
\]

Since here we assume that \( \ell \) is in limit circle case at one and in limit point case at the other endpoint the maximal operator \( L_{\text{max}} \) is the adjoint of a symmetric operator \( L \) with defect one. Its domain is given by
\[
\text{dom } L = \{ y \in \text{dom } L_{\text{max}} \lim_{x \to 0} W(y(x), g_+(x, \lambda_0)) = \lim_{x \to 0} W(y(x), g_-(x, \lambda_0)) = 0 \}.
\]

In this case \( g(\cdot, \lambda) \in L^2(0, \infty) \) is a defect element of \( L \). Note that, in fact, \( L \) does not depend on the particular choice of \( \lambda_0 \), cf. eg. [5].

**Remark 2.5.** In case that \( x = 0 \) is a regular endpoint the two expressions \( \lim_{x \to 0} W(y(x), g_+(x, \lambda_0)) \) and \( \lim_{x \to 0} W(y(x), g_-(x, \lambda_0)) \) can be written as \( y(0) \) and \( y'(0) \), respectively.

The self-adjoint extensions of \( L \) are given by \( L_\tau \) for \( \tau \in \mathbb{R} \) with
\[
\text{dom } L_\tau := \{ y \in \text{dom } L_{\text{max}} | \lim_{x \to 0} W(y(x), g_+(x, \lambda_0)) + \tau \lim_{x \to 0} W(y(x), g_-(x, \lambda_0)) = 0 \}
\]
and
\[
\text{dom } L_\infty := \{ y \in \text{dom } L_{\text{max}} | \lim_{x \to 0} W(y(x), g_-(x, \lambda_0)) = 0 \}.
\]

It is then a standard calculation to show that the following integral formulas for the resolvents hold:
\[
((L_\tau - \lambda)^{-1} y)(x) = -g(x, \lambda) \int_0^x g_+(s, \lambda) y(s) ds - g_+(s, \lambda) \int_x^\infty g(s, \lambda) y(s) ds,
\]
where \( g_+(\cdot, \lambda) = g_+(\cdot, \lambda) + \frac{\tau}{m(\lambda)} g_-(\cdot, \lambda) \) for \( \tau \in \mathbb{R} \) and \( g_+(\cdot, \lambda) = \frac{1}{m(\lambda)} g_-(\cdot, \lambda) \). Furthermore, from this one obtains directly
\[
(L_\tau - \lambda)^{-1} = (L_0 - \lambda)^{-1} - \frac{1}{m(\lambda) - \frac{\tau}{2}} (g(x, \lambda), \cdot) g(x, \lambda),
\]
where \( (\cdot, \cdot) \) denotes the usual inner product on \( L^2(0, \infty) \).

We introduce now the singular element \( \varphi := (L_0 - \lambda_0) g(\cdot, \lambda_0) \) for some \( \lambda_0 \in \mathbb{C} \). Note that \( \varphi \not\in L^2(0, \infty) \) since \( g \not\in \text{dom } L_0 \), however, \( \varphi \in \mathcal{H}_s(L_0) \).

Recall that if \( A \) is a semi bounded, self-adjoint linear operator in a Hilbert space \( \mathcal{H} \), \( A \geq \gamma \) for some \( \gamma \in \mathbb{R} \), then the scale of spaces \( \mathcal{H}_s(A) \) associated with \( A \) is defined as follows. For \( s \geq 0 \) the space \( \mathcal{H}_s(A) \) is given by the set \( \text{dom} (A - \mu)^{\frac{s}{2}} \) equipped with the norm
\[
\| y \|_s := \|(A - \mu)^{\frac{s}{2}} y \|_{\mathcal{H}}
\]
for some $\mu < \gamma$. However, it can easily be seen that this definition does not depend on $\mu$ and furthermore $\mathcal{H}_s(A)$ is complete with this norm. The space $\mathcal{H}_s(A)$ is then defined as the dual of $\mathcal{H}_s(A)$ (with respect to the original space $\mathcal{H}$), it can also be obtained by completing $\mathcal{H}$ with respect to the norm (2.16). This gives a scale of spaces $\mathcal{H}_s(A) \subset \mathcal{H}_{t}(A)$ if $s > t$. However, in this note we will deal with $s \in \mathbb{Z}$ only:

$$\begin{align*}
\text{dom}(A) & \subset \mathcal{H} \\
\mathcal{H} & \subset (\text{dom}(A))^* \\
\ldots & \subset \mathcal{H}_3(A) \subset \mathcal{H}_2(A) \subset \mathcal{H}_1(A) \subset \mathcal{H}_0(A) \subset \mathcal{H}_{-1}(A) \subset \mathcal{H}_{-2}(A) \subset \mathcal{H}_{-3}(A) \subset \ldots
\end{align*}$$

We are going to use the notation $\langle \cdot, \cdot \rangle$ not only for the usual inner product on the space $\mathcal{H}$, but $\langle g, f \rangle$ denotes also the action of the functional $f \in \mathcal{H}_{-s}(A)$ on an element $g \in \mathcal{H}_s(A)$, $s > 0$. Note that $(A - \mu)^{-\frac{s}{2}}$ can be seen as an isometry from $\mathcal{H}_s(A)$ to $\mathcal{H}_{s+t}$. In particular, $\langle g, (A - \mu)^{-\frac{s}{2}}f \rangle$ for $g \in \mathcal{H}_{s+t}$ and $f \in \mathcal{H}_{-s}$ is given by $\langle (A - \mu)^{-\frac{s}{2}}g, f \rangle$.

Let us now return to the differential operator $L_0$ in the limit point case. As already mentioned in the introduction, with the formal expression

$$L_t = L_0 + t\langle \varphi, \cdot \rangle \varphi \quad t \in \mathbb{R} \cup \{\infty\}$$

there is associated a family of self adjoint operators $L_\gamma$ given by

$$(L_\gamma - \lambda)^{-1} = (L_0 - \lambda)^{-1} - \frac{(L_0 - \lambda)^{-1} \varphi}{Q(\lambda) + \gamma} (L_0 - \lambda)^{-1} \varphi, \quad \gamma \in \mathbb{R} \cup \{\infty\},$$

where the $Q$-function is defined as $Q(\lambda) := (\lambda - \mu_1)(\varphi, (L_0 - \lambda)^{-1}(L_0 - \lambda)^{-1} \varphi)$ with some $\mu_1 \in \rho(L_0) \cap \mathbb{R}$.

In order to see the analogy in the next section we show here the well known fact, that $Q(\lambda) = m(\lambda) + c$ with some constant $c \in \mathbb{R}$. To this end with the above notation $Q$ can also be written as

$$Q(\lambda) = (\lambda - \mu_1)(g(\cdot, \lambda), g(\cdot, \mu_1)).$$

Using that $\ell(g(\cdot, \lambda)) = \lambda g(\cdot, \lambda)$ on every interval $[\varepsilon, \infty)$ for $\varepsilon > 0$, and then integrating by parts one obtains

$$Q(\lambda) = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty ((\ell - \mu_1)g(x, \lambda))g(x, \mu_1) \, dx$$

$$= \lim_{\varepsilon \to 0} W(g_-, \varepsilon, \lambda - m(\lambda)g_+(\varepsilon, \lambda), g_-(\varepsilon, \mu_1) - m(\mu_1)g_+(\varepsilon, \mu_1))$$

$$= m(\lambda) - m(\mu_1).$$

2.3. Limit point case. In what follows we concentrate on the case $q_0 \geq \frac{3}{2}$ only, that is $\ell$ is in limit point case at 0, or in other words, the operator $L_{\max}$ in (2.15) is self adjoint. Note that in this case every $y \in \text{dom} L_{\max}$ also satisfies the boundary condition $\lim_{x \to 0} W(y(x), g_+(x, \lambda_0)) = 0$. Hence without confusion here we can write $L_0$ instead of $L_{\max}$:

$$\text{dom} L_0 := \{y \in L^2(0, \infty) | y, y' \in AC_{\text{loc}}(0, \infty), \ell(y) \in L^2(0, \infty)\}$$

and

$$L_t = L_0 + t\langle \varphi, \cdot \rangle \varphi \quad t \in \mathbb{R} \cup \{\infty\}$$

Again we are interested in perturbations of the form

$$L_t = L_0 + t\langle \varphi, \cdot \rangle \varphi \quad t \in \mathbb{R} \cup \{\infty\}$$
with \( \varphi := (L_0 - \lambda_0)g(\cdot, \lambda_0) \), where \( g(\cdot, \lambda_0) \) is the function given in (2.7). However, in this case \( g(\cdot, \lambda_0) \) is not square integrable locally at 0, and hence we need some more considerations in order to make this definition precise and identify \( \varphi \) as an element from \( \mathcal{H}_n \) for a certain \( n \in \mathbb{N} \). As a first step the next lemma gives estimates for functions \( f \in \text{dom} \ L_0^k \).

**Lemma 2.6.** For \( f \in L^2(0, \infty) \) the resolvent of the operator \( L_0 \) is given by

\[
(2.19) \quad ((L_0 - \lambda)^{-1} f)(x) = -g(x, \lambda) \int_0^x g_+(s, \lambda) f(s) \, ds + g_+(x, \lambda) \int_x^\infty g(s, \lambda) f(s) \, ds.
\]

Let \( k \leq \left[ \frac{3 + \sqrt{3} + \Phi_0}{2} \right] \), then for every \( f \in \text{dom} \ L_0^k \) there exists a constant \( C > 0 \) such that for some \( x_0 > 0 \) it holds:

\[
(2.20) \quad |f(x)| \leq C x^{-\frac{1}{2} + 2k} \quad \forall x \in (0, x_0),
\]

\[
(2.21) \quad |f'(x)| \leq C x^{-\frac{1}{2} + 2k} \quad \forall x \in (0, x_0).
\]

**Proof.** For \( \lambda \in \rho(L_0) \) let us denote by \( R(\lambda)f \) the integral on the right side of (2.19), which is well defined for every function \( f \in L^2(0, \infty) \). We first show (2.19) for \( f \in C_0^\infty((0, \infty)) \). In particular, we have to show that \( R(\lambda)f \) is square integrable locally at 0 and \( \infty \), if \( f \) has compact support in \( (0, \infty) \). Note

\[
(R(\lambda)f)(x) = -g_+(x, \lambda) \int_{\text{supp} f} g(s, \lambda) f(s) \, ds \quad \text{for } x < x_0 < \min \text{ supp} f
\]

and

\[
(R(\lambda)f)(x) = -g(x, \lambda) \int_{\text{supp} f} g_+(s, \lambda) f(s) \, ds \quad \text{for } \max \text{ supp} f < x_1 < x.
\]

This in particular shows \( R(\lambda)f|_{(0, x_0)} \in L^2(0, x_0) \) and \( R(\lambda)f|_{(x_1, \infty)} \in L^2(x_1, \infty) \).

Using Proposition 2.1 (iii) it is straightforward to see \( \ell(R(\lambda)f) = \lambda R(\lambda)f \) and hence \( R(\lambda)f = (L_0 - \lambda)^{-1} f \). For an arbitrary \( f \in L^2(0, \infty) \) there exists a sequence \( f_n \in C_0^\infty((0, \infty)) \) such that \( \|f - f_n\|_{L^2} \to 0 \). Since the resolvent \( (L_0 - \lambda)^{-1} \) is a bounded operator it holds

\[
(L_0 - \lambda)^{-1} f = L_2 \lim_{n \to \infty} (L_0 - \lambda)^{-1} f_n = L_2 \lim_{n \to \infty} R(\lambda)f_n.
\]

Since for every \( x \in (0, \infty) \) the continuous functions \( (R(\lambda)f_n)(x) \) converge to the continuous function \( (R(\lambda)f)(x) \), this pointwise limit coincides with the \( L^2 \)-limit, which finally gives

\[
(L_0 - \lambda)^{-1} f = R(\lambda)f \quad \text{for every } f \in L^2(0, \infty).
\]

In order to show (2.20) we use induction. The asymptotic expansion (2.3) for \( g_+ \) and Cauchy-Schwarz-inequality imply for \( x \in (0, x_0) \) with some fixed \( x_0 \)

\[
|g(x, \lambda) \int_0^x g_+(s, \lambda) f(s) \, ds| \leq C_1 x^\alpha - \int_0^x s^{\alpha + \frac{1}{2}} |f(s)| \, ds \leq C_2 x^{\alpha + \frac{1}{2}} = C_2 x^\frac{3}{2}
\]

(2.22)
and

\[
|g_+(s, \lambda)\int_x^\infty g(s, \lambda)f(s)\,ds| \leq C_3x^{\alpha+\varepsilon}\left(\int_x^{x_0} s^{\alpha-1}|f(s)|\,ds + \int_{x_0}^\infty |g(s, \lambda)f(s)|\,ds\right)
\]

(2.23)

\[
\leq C_4x^{\alpha+\alpha+\varepsilon+\frac{1}{2}} + C_5x^{\alpha-1} \leq C_6x^{\frac{3}{2}},
\]

where we have used that if \(q_0 \geq \frac{3}{4}\) then \(\alpha_+ \geq \frac{3}{2}\). Assume now \(f \in \text{dom } L_0^{k+1}\), that is \(f = (L_0 - \lambda)^{-1}h\) with some \(h \in \text{dom } L_0^k\) and \(\lambda \in \mathbb{R}^-\). Then using (2.20) for \(h\) one obtains the corresponding estimate as in (2.22) and noting \(-\frac{1}{2} + 2k \leq \alpha_+\) also (2.23). This proves (2.20) for \(k + 1\). In the same way (2.21) can be shown. \(\square\)

The following theorem establishes the connection between the functions \(g(\cdot, \lambda)\) and \(g_k\) defined in (2.7) and (2.8), respectively, and the operator \(L_0\). In particular, it makes the definition of \(\varphi := (L_0 - \lambda_0)g(\cdot, \lambda_0)\) precise and hence will enable us to apply the theory of singular perturbations.

**Theorem 2.7.** Let \(g\) and \(g_k\) be given as above and \(n = 2 + \left[\sqrt{\frac{1}{4} + q_0}\right]\). Then the element \(\varphi := (L_0 - \lambda_0)g(\cdot, \lambda_0)\) is independent of the particular choice of \(\lambda_0 \in \mathbb{R}^+\) with \(L_0 - \lambda_0 > 0\) and

\[\varphi \in \mathcal{H}_{-n}(L_0) \setminus \mathcal{H}_{-n+1}(L_0).\]

Furthermore, it holds

\[g_k = (L_0 - \mu_k)^{-1} \ldots (L_0 - \mu_1)^{-1}\varphi\]

(2.24)

and, in particular,

\[g_k \in \mathcal{H}_{-n+2k}(L_0) \setminus \mathcal{H}_{-n+2k+1}(L_0).\]

(2.25)

**Remark 2.8.** In case \(q_1 = 0\) and \(\alpha_+ - \alpha_- \notin \mathbb{N}_{\text{even}}\) in [4] a modified Hankel transform is applied to the problem, and then these statements become obvious. However, this transformation makes essential use of well known properties of Bessel functions and hence in the general case we have to prove the theorem differently.

**Proof.** Since we consider the scale of Hilbert spaces only corresponding to the operator \(L_0\), within this proof we are going to write simply \(\mathcal{H}_s\) instead of \(\mathcal{H}_s(L_0)\). Lemma 2.3 implies that for some large enough index \(m\) the function \(g_m\) belongs to \(\mathcal{H}_0 \setminus \mathcal{H}_2\). In order to determine the number \(m\) note that the latter is equivalent to \(g_m \in L^2(0, \infty)\) but \(\ell(g_m) \not\in L^2(0, \infty)\). Since \(\ell(g_m) = \mu_m g_m + g_{m-1}\) the asymptotic expansion (2.11) implies that this is further equivalent to

\[2(\alpha_- + 2(m - 1)) > 1 \land 2(\alpha_- + 2(m - 1) - 2) \leq -1,
\]

from which we can conclude

\[m = \left[\frac{3 + \sqrt{\frac{1}{4} + q_0}}{2}\right].\]

(2.26)

In the next step we show that for \(k = 1, \ldots, m\) it holds

\[g_k \in \mathcal{H}_{-2(m-k)} \setminus \mathcal{H}_{-2(m-k)+2}\]

(2.27)

and

\[(L_0 - \mu_k)g_k = g_{k-1}.
\]

(2.28)
Consider first $k = m$. Then we already have $g_m \in \mathcal{H}_0 \setminus \mathcal{H}_2$. Hence $(L_0 - \mu_m)g_m$ is an element from $\mathcal{H}_{-2}$ and we are going to show that, in fact, it coincides with the function $g_{m-1}$. To this end apply $(L_0 - \mu_m)g_m$ to an arbitrary $f \in \mathcal{H}_2$, that is
\[
(f, (L_0 - \mu_m)g_m) = \int_0^\infty g_m(x)((L_0 - \mu_m)f)(x) \, dx
\]
and the integral term have an expansion starting with (2.11). Inserting the asymptotic expansion (2.11) one sees that both the boundary term (2.29) $\lim_{\varepsilon \to 0} \int_\varepsilon^\infty g_m(x)(\ell - \mu_m)f(x) \, dx$.

By integration by parts this becomes
\[
\lim_{\varepsilon \to 0} W(f(\varepsilon), g_m(\varepsilon)) + \int_\varepsilon^\infty (\ell - \mu_m)g_m(x)f(x) \, dx.
\]

Here the first limit exists according to Lemma 2.6 and equals 0. Note that for every $k \geq 2$ it holds
\[
(\ell - \mu_k)g_k(x) = g_{k-1}(x) \quad x \in (0, \infty).
\]

This finally gives
\[
(f, (L_0 - \mu_m)g_m) = \int_0^\infty g_{m-1}(x)f(x) \, dx,
\]
this (2.28) for $k = m$. Next we reduce the number $k$ step by step. Assume now that the relations (2.27) and (2.28) already hold for some $k > 1$. Then (2.28), in particular, implies $g_{k-1} \in \mathcal{H}_{-2(m-k-2)} \setminus \mathcal{H}_{-2(m-k)}$. Take now an arbitrary function $f \in \mathcal{H}_{2(m-k)+2} = \text{dom } L_{-2(m-k)}$ and consider $\langle f, (L_0 - \mu_{k-1})g_{k-1} \rangle$ as above. Then (2.28) for $k - 1$ follows in the same way, where again the estimates (2.20) and (2.21) are essential and (2.24) is proved for $k \leq m$. We leave the details to the reader.

Since we know now that, in particular, $g(\cdot, \mu_1) \in \mathcal{H}_{-2m+2} \setminus \mathcal{H}_{-2m+4}$ the element $\varphi := (L_0 - \mu_1)g(\cdot, \mu_1)$ is well defined and belongs to $\mathcal{H}_n$ for $n$ either $2m-1$ or $2m$. In the next step of the proof we are going to determine $n$ precisely. Obviously $\varphi \in \mathcal{H}_{-2m+1} \setminus \mathcal{H}_{-2m+2}$ exactly if $g_m \in \mathcal{H}_1 \setminus \mathcal{H}_2$, that is, $g_m$ belongs to the domain of the quadratic form associated with the operator $L_0$, or in other words, this happens if and only if the following integral converges:
\[
\int_0^\infty \left( |g_m'(x)|^2 + \frac{q_0 + q_1x}{x^2} |g_m(x)|^2 \right) \, dx,
\]

By integration by parts this becomes
\[
\lim_{\varepsilon \to 0} -g_m'(\varepsilon)g_m(\varepsilon) + \int_\varepsilon^\infty \ell(g_m(x))g_m(x) \, dx.
\]

Inserting the asymptotic expansion (2.11) one sees that both the boundary term and the integral term have an expansion starting with $\varepsilon^{2(\alpha_+ + 2(m-1))-1}$. The only exception here is if $2(\alpha_+ + 2(m - 1)) = 0$, then the integral starts with a logarithmic term. Hence if $2(\alpha_+ + 2(m - 1)) < 0$ the limit (2.29) exists, if however $2(\alpha_+ + 2(m - 1)) > 0$ we have to investigate the leading coefficient, which equals
\[
-(\alpha_+ + 2(m - 1)) \left( \frac{C_{m-1}(m)}{C_{2(m-1)}(m-1)} \right)^2 = \frac{C_{2(m-1)}(m)}{2(\alpha_+ + 2(m - 1))} - 1.
\]
Inserting here the recursion for the coefficients $C_{2(k-1)}^{(k)}$ from Remark 2.4 this further equals
\[
\left( \frac{C_{2(m-1)}^{(m)}}{C_{2(m-1)}} \right)^2 - \frac{(\alpha_- + 2(m-1))(2(\alpha_- + 2(m-1)) - 1) + 2(m-1)\alpha_- + 2m - 3}{2(\alpha_- + 2(m-1)) - 1},
\]
where the numerator can further be simplified to
\[
(\alpha_- - 2(m - 1))^2 + \alpha_- (\alpha_- - 1).
\]
Since in this section we assumed limit circle case $\alpha_-(\alpha_- - 1) > 0$ and thus (2.31) can not vanish. Hence the limit in (2.29), indeed, exists if and only if the inequality $2(\alpha_- + 2(m-1)) - 1 > 0$ holds. Inserting the formula (2.26) for $m$ one easily finds that this inequality is satisfied if and only if $\left[ \sqrt{\frac{1}{4} + q_0} \right]$ is an odd number. In this case $\varphi \in \mathcal{H}_{-2m-1} \setminus \mathcal{H}_{-2m+2}$ and $2m - 1$ can be written as $2m - 1 = \left[ \sqrt{\frac{1}{4} + q_0} \right] + 2$.

In the other case, however, $\varphi \in \mathcal{H}_{-2m} \setminus \mathcal{H}_{-2m+1}$ and $2m = \left[ \sqrt{\frac{1}{4} + q_0} \right] + 2$. Hence in both cases it holds
\[
\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1} \quad \text{for} \quad n = \left[ \sqrt{\frac{1}{4} + q_0} \right] + 2.
\]

We show now that $\varphi := (L_0 - \mu_1)g(\cdot, \mu_1)$ is independent of the particular choice of $\mu_1$. To this end apply $\varphi$ to $f \in \mathcal{H}_q$. Integration by parts gives
\[
\langle f, \varphi \rangle = \langle (L_0 - \mu_1)f, g(\cdot, \mu_1) \rangle = \int_{0}^{\infty} g(x, \mu_1)(\ell - \mu_1)f(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} W(g(\varepsilon, \mu_1), f(\varepsilon)).
\]

Since $g(\varepsilon, \mu_1) = \frac{1}{\alpha_- - \alpha_+} \varepsilon^{\alpha_-} + o(\varepsilon^{\alpha_- + 1})$ and $g'(\varepsilon, \mu_1) = \frac{\alpha_-}{\alpha_- - \alpha_+} \varepsilon^{\alpha_- - 1} + o(\varepsilon^{\alpha_-})$

Lemma 2.6 implies
\[
\lim_{\varepsilon \to 0} W(g(\varepsilon, \mu_1), f(\varepsilon)) = \frac{1}{\alpha_- - \alpha_+} \lim_{\varepsilon \to 0} W(\varepsilon^{\alpha_-}, f(\varepsilon)),
\]
which indeed is independent of the point $\mu_1$.

Finally (2.28), or equivalently
\[
(L_0 - \mu_k)^{-1} g_{k-1} = g_k
\]
follows also for $k > m$ directly by using the defining relation (2.8) for $g_{k-1}$ and applying the resolvent equation to $(L_0 - \mu_k)^{-1}(L_0 - \mu_i)^{-1} \varphi$.

\[\square\]

\textbf{Remark 2.2.} As we have seen $\varphi$ is not a usual function (as the $g_i$’s for $i \geq 1$), but a distribution with support at the point $x = 0$ only:
\[
\langle \varphi, f \rangle = \frac{1}{\alpha_- - \alpha_+} \lim_{x \to 0} W(x^{\alpha_-}, f(x)).
\]

3. A Family of Operators in the Limit Point Case

In this section the model for the singular perturbation (2.18) is studied and the connection to the generalized Titchmarsh-Weyl-coefficient $m$ is explored.
3.1. Cascade representation of the model. Let us first recall (see [8, 3]) a model for the operator family which formally is given by the following expression

\[ A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \]

where \( A \) is a self-adjoint semi-bounded linear operator acting in a Hilbert space \( \mathcal{H} \) and \( \varphi \in \mathcal{H}_{-n}(A) \setminus \mathcal{H}_{-n+1}(A) \) is a singular element. In the case that \( \varphi \) does not belong to \( \mathcal{H}_{-2}(A) \) the corresponding perturbation is said to be super-singular and cannot be defined in the original space \( \mathcal{H} \). In what follows we are going to describe the model for (3.1) as a self-adjoint operator in the form presented first in [9, 8] and modified later in [3]. Developing this model one faced two at the first glance incompatible requirements:

- The restriction \( A |_{\{\psi \in \text{dom}(A) : \langle \psi, \varphi \rangle = 0\}} \) is essentially self-adjoint unless it is considered in the Hilbert space \( \mathcal{H}_{-n-2}(A) \) where the operator \( A \) is defined on the domain \( \mathcal{H}_{n}(A) \);
- The formal resolvent of \( A_\alpha \), suggested by Krein’s formula, contains elements of the form \( (A - \mu)^{-1} \varphi \in \mathcal{H}_{-n+2}(A) \setminus \mathcal{H}_{-n+3}(A) \), which do not even belong to the space \( \mathcal{H}_0 \).

In order to meet both requirements it was suggested to define models for \( A_\alpha \) in the extended Hilbert space \( \mathbb{H} := \mathbb{C}^{n-2} \oplus \mathcal{H}_{-n-2}(A) \) containing elements \( \mathbf{U} := (\bar{u}, U) \), with \( \bar{u} \in \mathbb{C}^{n-2}, U \in \mathcal{H}_{n-2}(A) \) and equipped with the scalar product

\[ \langle \mathbf{U}, \mathbf{V} \rangle_{\mathbb{H}} := \langle \bar{u}, \bar{v} \rangle_{\mathbb{C}^{n-2}} + \langle U, b_{n-2}(A)V \rangle \quad \text{for } \mathbf{U}, \mathbf{V} \in \mathbb{H} = \mathbb{C}^{n-2} \oplus \mathcal{H}_{n-2}(A), \]

where \( \Gamma \) is a certain Gram matrix (to be specified later) and \( b_{n-2} \) denotes the regularizing polynomial, that is,

\[ b_{n-2}(\lambda) := (\lambda - \mu_1)(\lambda - \mu_2) \ldots (\lambda - \mu_{n-2}), \]

where \( \mu_i \in \rho(A) \) such that \( A - \mu_i > 0 \). Note that the norm determined by \( \langle \mathbf{U}, b_{n-2}(A)V \rangle \) is equivalent to the standard norm in the space \( \mathcal{H}_{n-2}(A) \) given by \( \langle \mathbf{U}, (A - \mu)^{-n-2}V \rangle \), where \( \mu \) is a negative number from the resolvent set of \( L \).

Every element \( \mathbf{U} \) from \( \mathbb{H} \) can be viewed as an element from \( \mathcal{H}_{-n+2}(A) \) using the following natural embedding

\[ \rho \mathbf{U} := \sum_{j=1}^{n-2} u_j g_j + U, \]

where we used the following notation

\[ g_j := (A - \mu_j)^{-1} \ldots (A - \mu_1)^{-1} \varphi. \]

Note that the restriction \( A_{\text{min}} := A |_{\{U \in \mathcal{H}_{n-2}(A) : \langle \varphi, U \rangle = 0\}} \) as an operator in \( \mathcal{H}_{n-2}(A) \) is symmetric with defect (1, 1). Consider \( A_{\text{max}} := A_{\text{min}}{^\dagger} \) - the triplet adjoint with respect to the triple \( \mathcal{H}_{n-2}(A) \subset \mathcal{H}_0(A) \subset \mathcal{H}_{-n+2}(A) \), cf. [2]. Then the maximal operator \( A_{\text{max}} \) acting in \( \mathbb{H} \) is defined by

\[ \mathcal{H}_{\text{max}} := \rho^{-1} A_{\text{max}} \rho. \]

From this one can show that its domain is given by

\[ \text{dom} (\mathcal{H}_{\text{max}}) = \{ \mathbf{U} = (\bar{u}, U) \in \mathbb{H} : \bar{u} \in \mathbb{C}^{n-2}, \]

\[ U = u_{n-1} g_{n-1} + U_r, u_{n-1} \in \mathbb{C}, U_r \in \mathcal{H}_{n}(A) \} \]
and it acts as

\begin{align}
\mathbb{A}_{\text{max}} \mathbf{U} = \mathbb{A}_{\text{max}} \begin{pmatrix} \vec{u} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} M \vec{u} + u_{n-1} \vec{e}_{n-2} \\ AU_r + \mu_{n-1} u_{n-1} g_{n-1} \end{pmatrix},
\end{align}

with the basis vector \( \vec{e}_{n-2} = (0, ..., 0, 1) \) and the \((n-2) \times (n-2)\) matrix \( M \) given by

\begin{equation}
M := \begin{pmatrix}
\mu_1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \mu_2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \mu_3 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_4 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \mu_{n-4} & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & \mu_{n-3} & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_{n-2}
\end{pmatrix}.
\end{equation}

Note that alternatively the operator \( \mathbb{A}_{\text{max}} \) can also be determined uniquely by the condition

\[ \rho \mathbb{A}_{\text{max}} \equiv \mathbb{A} \rho. \]

The self adjoint operators corresponding to the formal expression (3.1) are then constructed as restrictions of \( \mathbb{A}_{\text{max}} \). Obviously this is possible only if \( \mathbb{A}_{\text{max}} \) is the adjoint of a symmetric operator. It has been shown that this is the case only if the Gram matrix \( \Gamma \) satisfies

\[ \Gamma M = M^* \Gamma = 0, \]

i.e., if the matrix \( M \) is Hermitian with respect to the scalar product given by the Gram matrix \( \Gamma \): \( \langle M \vec{u}, \Gamma \vec{v} \rangle_{C^{n-2}} = \langle \vec{u}, \Gamma M \vec{v} \rangle_{C^{n-2}} \). It turned out that if all \( \mu_j \), for \( j = 1, 2, ..., n-2 \) are mutually distinct, then there exists an \( n-2 \)-parameter family of positive Hermitian matrices \( \Gamma \) satisfying (3.8) (see [3] for details). Note that if at least two of the \( \mu_j \) are equal, then every matrix \( \Gamma \) solving (3.8) is indefinite, which implies that the corresponding space \( \mathcal{H} \) is not a Hilbert space. Hence in what follows we assume that all \( \mu_j \) for \( j = 1, 2, ..., n-2 \) are distinct and \( \Gamma \) is a positive Hermitian matrix satisfying (3.8).

All the self adjoint restrictions of \( \mathbb{A}_{\text{max}} \) can then be characterized by the generalized "boundary conditions"

\[ \cos \theta \ u_{n-1} + \sin \theta \left( \langle \varphi, U_r \rangle - \langle \vec{e}_{n-2}, \Gamma \vec{u} \rangle_{C^{n-2}} \right) = 0. \]

Such a condition describes a linear relation between the vector \( \vec{u} \), the coefficient \( u_{n-1} \) and the function \( U_r \in \mathcal{H}(A) \), or more precisely \( \langle \varphi, U_r \rangle \), note here \( \varphi \in \mathcal{H}_{-n}(A) \) and \( U_r \in \mathcal{H}_n(A) \).

So the operator \( \mathbb{A}_\theta \) is given as follows.

**Def 3.1.** For \( \theta \in [0, \pi) \) the operator \( \mathbb{A}_\theta \) is defined on the domain

\begin{align}
\text{dom} (\mathbb{A}_\theta) := \{ \mathbf{U} = (\vec{u}, \mathbf{U}) \in \mathbb{H} : \\
U = u_{n-1} g_{n-1} + U_r, u_{n-1} \in \mathbb{C}, U_r \in \mathcal{H}(A) \\
\cos \theta \ u_{n-1} + \sin \theta \left( \langle \varphi, U_r \rangle - \langle \vec{e}_{n-2}, \Gamma \vec{u} \rangle_{C^{n-2}} \right) = 0 \},
\end{align}

acting as

\begin{align}
\mathbb{A}_\theta \left( \begin{pmatrix} \vec{u} \\ \mathbf{U} \end{pmatrix} \right) = \mathbb{A}_\theta \left( \begin{pmatrix} \vec{u} \\ U_r + u_{n-1} g_{n-1} \end{pmatrix} \right) := \begin{pmatrix} M \vec{u} + u_{n-1} \vec{e}_{n-2} \\ AU_r + \mu_{n-1} u_{n-1} g_{n-1} \end{pmatrix}.
\end{align}
For completeness and since some of the formulas to be derived will be needed later on we are showing now that the model operators $\Lambda_\theta$ are self adjoint.

**Proposition 3.2.** The operator $\Lambda_\theta$ given by Definition 3.1 is self adjoint in the Hilbert space $\mathbb{H}$, provided $\Gamma$ satisfies (3.8).

**Proof.** We are going to prove that $\Lambda_\theta$ is symmetric and for non-real $\lambda$ its resolvent is a bounded operator defined on the whole space $\mathbb{H}$. As to the symmetry, a straightforward calculation with taking into account relation (3.8) and the obvious identity $(A - \mu_{n-1})b_{n-1}(A)g_{n-1} = \varphi$ gives the following expression for the boundary form

\begin{equation}
\langle U, \mathbb{L}_\theta V \rangle_{\mathbb{H}} = \langle L_\theta U, V \rangle_{\mathbb{H}} = \langle \tilde{e}_{n-2}, \Gamma \tilde{u} \rangle_{\mathbb{C}^{n-2}} - \langle \varphi, U_r \rangle_{\mathbb{C}^{n-2}} - \langle \varphi, V_r \rangle_{\mathbb{C}^{n-2}},
\end{equation}

which vanishes since both elements $U = (\tilde{u}, U_r)$ and $V = (\tilde{v}, V)$ satisfy the boundary condition in (3.10).

Let now $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We show that then for every $F = \begin{pmatrix} \tilde{f} \\ F \end{pmatrix} \in \mathbb{H}$ the equation

\begin{equation}
(L_\theta - \lambda) U = F
\end{equation}

has a unique solution. With $U = (\tilde{u}, U_r + u_{n-1}g_{n-1})$ equation (3.13) can be written as the system of linear equations

\begin{equation}
\begin{array}{l}
(M - \lambda)\tilde{u} + u_{n-1}\tilde{e}_{n-2} = \tilde{f}, \\
(A - \lambda)U_r + (\mu_{n-1} - \lambda)u_{n-1}g_{n-1} = F,
\end{array}
\end{equation}

\begin{equation}
\cot \theta u_{n-1} + \langle \varphi, U_r \rangle - \langle \tilde{e}_{n-2}, \Gamma \tilde{u} \rangle_{\mathbb{C}^{n-2}} = 0,
\end{equation}

where the last equation is just (3.9). Since $M$ and $L$ are self adjoint in Hilbert spaces we can apply the the resolvents of $M$ and $L$, respectively, to the first and second equations and get

\begin{equation}
\begin{array}{l}
\tilde{u} = (M - \lambda)^{-1}\tilde{f} - u_{n-1}(M - \lambda)^{-1}\tilde{e}_{n-2} \\
U_r = (A - \lambda)^{-1}F + u_{n-1}(A - \mu_{n-1})(A - \lambda)^{-1}g_{n-1}.
\end{array}
\end{equation}

Inserting this in the third equation yields

\begin{equation}
u_{n-1} = -\frac{\langle \varphi, (A - \lambda)^{-1}F \rangle - \langle \tilde{e}_{n-2}, \Gamma (M - \lambda)^{-1}\tilde{f} \rangle_{\mathbb{C}^{n-2}}}{QA(\lambda) + QM(\lambda) + \cot \theta},
\end{equation}

where the $Q$-functions corresponding to the operators $A$ and $M$ have been denoted as

\begin{equation}Q_A(\lambda) := \langle \varphi, (\lambda - \mu_{n-1})(A - \lambda)^{-1}g_{n-1} \rangle = \langle g_{n-2}, b_{n-2}(\lambda - \mu_{n-1})(A - \lambda)^{-1}(A - \mu_{n-1})^{-1}g_{n-2} \rangle,
\end{equation}

\begin{equation}Q_M(\lambda) := \langle \tilde{e}_{n-2}, \Gamma (M - \lambda)^{-1}\tilde{e}_{n-2} \rangle.
\end{equation}

Again (3.15) finally implies

\begin{equation}\tilde{u} = (M - \lambda)^{-1}\tilde{f} - \frac{\langle \varphi, (A - \lambda)^{-1}F \rangle - \langle \tilde{e}_{n-2}, \Gamma (M - \lambda)^{-1}\tilde{f} \rangle_{\mathbb{C}^{n-2}}}{QL(\lambda) + QM(\lambda) + \cot \theta}(M - \lambda)\tilde{e}_{n-2},
\end{equation}

\begin{equation}U = (A - \lambda)^{-1}F - \frac{\langle \varphi, (A - \lambda)^{-1}F \rangle - \langle \tilde{e}_{n-2}, \Gamma (M - \lambda)^{-1}\tilde{f} \rangle_{\mathbb{C}^{n-2}}}{QL(\lambda) + QM(\lambda) + \cot \theta}(A - \lambda)^{-1}g_{n-2},
\end{equation}
and hence we have shown directly that the equation (3.13) is solvable for arbitrary non-real \( \lambda \). Thus \( \mathbb{A}_\theta \) is self adjoint. \hfill \Box

In order to write the resolvent \((\mathbb{A}_\theta - \lambda)^{-1}\) and the restricted-embedded resolvent \(\varrho(\mathbb{A}_\theta - \lambda)^{-1}|_{\mathcal{H}_{n-2}}\) in the form of Krein’s formula we introduce the vector

\[
\Phi(\lambda) := \left( (M - \lambda)^{-1} \bar{c}_{n-2}, (A - \lambda)^{-1} g_{n-2} \right) \in \mathbb{H},
\]

which is in fact a deficiency element for the operator \( \mathbb{A}_{\min} = \mathbb{A}_{\max}^* \). Furthermore, put

\[
Q(\lambda) := Q_A(\lambda) + Q_M(\lambda) = (\lambda - \mu_{n-1}) (\Phi(\mu_{n-1}), \Phi(\lambda))_\mathbb{H} + \langle \bar{c}_{n-2}, \Gamma(M - \mu_{n-1})^{-1} e_{n-2} \rangle_{\mathcal{C}_{n-2}}.
\]

Note that \( Q_A \) and \( Q_M \) are both Nevanlinna functions and hence also \( Q \) is a Nevanlinna function, \( Q_M \) is even rational.

**Proposition 3.3.** Let the self-adjoint operators \( A \) and \( \mathbb{A}_\theta \), the defect element \( \Phi \) and the function \( Q \) be related as above. Then it holds

\[
(\mathbb{A}_\theta - \lambda)^{-1} = (A - \lambda)^{-1} - \frac{1}{Q(\lambda) + \cot \theta} \langle \Phi(\lambda), \cdot \rangle_\mathbb{H} \Phi(\lambda),
\]

where \( \mathbb{A}_\theta \) is the operator corresponding to the parameter \( \theta = 0 \), that is, \( \mathbb{A}_\theta = M \oplus L \). Moreover, with the natural embedding \( \varrho : \mathbb{H} \to \mathcal{H}_{-n+2} \), given by (3.4), it holds

\[
\rho(\mathbb{A}_\theta - \lambda)^{-1}|_{\mathcal{H}_{n-2}} = (A - \lambda)^{-1} - \frac{1}{b_{n-2}(\lambda)(Q(\lambda) + \cot \theta)} \langle (A - \lambda - \lambda) - 1 \varphi, \cdot \rangle (A - \lambda)^{-1} \varphi,
\]

with the polynomial \( b_{n-2} \) defined in (3.3).

**Remark 3.4.** Note that in the restricted-embedded resolvent formula (3.18) the function

\[
d(\lambda) := b_{n-2}(\lambda)(Q(\lambda) + \cot \theta)
\]

is a generalized Nevanlinna function, which obviously can also be written as

\[
d(\lambda) = b_{n-2}(\lambda)Q_A(\lambda) + p_{n-2}(\lambda) + \cot \theta b_{n-2}(\lambda),
\]

where \( p_{n-2}(\lambda) \) denotes the polynomial \( b_{n-2}(\lambda)Q_M(\lambda) \).

**Proof.** Writing the solution of equation (3.13), which was given explicitly in the proof of Proposition 3.2, with the help of the element \( \Phi \) gives directly the first resolvent formula (3.17). For (3.18) we need the embedding of the defect element

\[
\varrho \Phi = \frac{1}{b_{n-2}(\lambda)} (A - \lambda)^{-1} \varphi.
\]

To this end note that

\[
\rho(M - \lambda)^{-1} e_{n-2} = \frac{1}{b_{n-2}(\lambda)} \sum_{j=1}^{n-2} b_{j-1}(\lambda)b_j(A)^{-1} \varphi = \frac{1}{b_{n-2}(\lambda)} (A - \lambda)^{-1} [\varphi - b_{n-2}(\lambda)b_{n-2}(A)^{-1} \varphi],
\]

where the second equality results from the identity

\[
b_{j-1}(\lambda)b_j(A)^{-1} \varphi = (A - \lambda)^{-1} [b_{j-1}(\lambda)b_{j-1}(A)^{-1} \varphi - b_j(\lambda)b_j(A)^{-1} \varphi].
\]

Then formula (3.18) follows immediately. \hfill \Box
3.2. **On the connection between the functions** $m$ **and** $d$. We are now coming back to the particular differential operator $L_0$ given in (2.17). In Theorem 2.7 we have shown that the distribution $\varphi = (L_0 - \lambda_0)g(\cdot, \lambda_0)$ belongs to $\mathcal{H}^{-n}(L_0)$ and hence the operator with the super singular perturbation $L_0 + \alpha \langle \varphi, \cdot \rangle \varphi$

gives rise to the model described in the preceding section. That is, the model space is $\mathbb{H} := \mathbb{C} \oplus \mathcal{H}_{n-2}(L_0)$ with the inner product

$$(U, V)_{\mathbb{H}} := (\bar{u}, \Gamma \bar{v})_{\mathbb{C}^n} + \langle U, b_{n-2}(L_0)V \rangle \quad \text{for} \quad U, V \in \mathbb{H} = \mathbb{C} \oplus \mathcal{H}_{n-2}(L_0).$$

Note that here the range of the embedding $\varrho : \mathbb{H} \to \mathcal{H}^{-n+2}(L_0)$ consists of functions only, which may have non square-integrable singularities only at $x = 0$. The self adjoint model operators are given as in Definition 3.1 and, in particular, Proposition 3.3 for the resolvent and the restricted-embedded resolvent holds.

The main result of this paper is now the following link between the generalized Titchmarsh-Weyl coefficient $m$ (in Fulton’s analytic approach) and $d$ (in the above singular perturbation approach). Recall, that in the regular case $q_0 < \frac{3}{4}$ the Titchmarsh-Weyl-coefficient and the Q-function differ only by a constant.

**Theorem 3.1.** Let $m$ be the generalized Titchmarsh-Weyl-coefficient in (2.7) and $d$ the generalized Nevanlinna function in (3.19). Then it holds

$$d(\lambda) - m(\lambda) = \delta(\lambda),$$

where $\delta$ is a polynomial of degree $\leq n - 2 = \left\lfloor \sqrt{\frac{3}{4}} + q_0 \right\rfloor$.

**Proof.** By integrating by parts the first summand of $d$ can be rewritten as

$$b_{n-2}(\lambda)(g_{n-1}(\cdot), (\lambda - \mu_{n-1})g(\cdot, \lambda)) =$$

$$= \lim_{\varepsilon \to 0} b_{n-2}(\lambda) \int_{\varepsilon}^{\infty} g_{n-1}(x)(\ell - \mu_{n-1} - (\ell - \lambda))g(x, \lambda) \, dx$$

$$= \lim_{\varepsilon \to 0} (\lambda - \mu_1) \ldots (\lambda - \mu_{n-2}) \left[ - g_{n-1}(\cdot)g'(\cdot, \lambda) + g'_{n-1}(\cdot)g(\cdot, \lambda) \right]_{\varepsilon}^{\infty}$$

$$+ \int_{\varepsilon}^{\infty} g_{n-2}(x)g(x, \lambda) \, dx,$$

and repeating this calculation with each factor $(\lambda - \mu_i)$ leads to

$$(3.20) \quad = \lim_{\varepsilon \to 0} W(G(\varepsilon, \lambda), g(\varepsilon, \lambda))$$

with $G(\cdot, \lambda) := \sum_{i=1}^{n-1} \left( \prod_{j=1}^{i-1} (\lambda - \mu_j) \right) g_i(\cdot)$. According to (2.8) this function can be written as

$$G(\cdot, \lambda) = \sum_{i=1}^{n-1} R_i(\lambda)g_i(\cdot)$$

with polynomials

$$R_i(\lambda) := \sum_{k=i}^{n-1} b_{k-1}(\lambda)A_i^{(k)},$$
where the coefficients $A^{(k)}_i$ were defined in (2.9) and we set $b_0(\lambda) \equiv 0$. Note, in particular, the $R_i$‘s are of degree $n - 2$ and $\sum_{i=1}^{n-1} R_i(\lambda) = 1$. Expanding (3.20) using (2.7) gives

$$\lim_{\varepsilon \to 0} W(G(\varepsilon, \lambda), g(\varepsilon, \lambda)) =$$

$$= \lim_{\varepsilon \to 0} W\left(\sum_{i=1}^{n-1} R_i(\lambda)g_-(\varepsilon, \mu_i), g_-(\varepsilon, \lambda)\right)$$

$$= -\sum_{i=1}^{n-1} R_i(\lambda)m(\mu_i) \lim_{\varepsilon \to 0} W(g_+(\varepsilon, \mu_i), g_-(\varepsilon, \lambda))$$

$$= -m(\lambda) \lim_{\varepsilon \to 0} W\left(\sum_{i=1}^{n-1} R_i(\lambda)g_-(\varepsilon, \mu_i), g_+(\varepsilon, \lambda)\right)$$

$$= +m(\lambda) \sum_{i=1}^{n-1} R_i(\lambda)m(\mu_i) \lim_{\varepsilon \to 0} W(g_+(\varepsilon, \mu_i), g_+(\varepsilon, \lambda))$$

According to Proposition 2.1 the limits in (3.22), (3.23), and (3.24) are $1$, $-1$, and $0$, respectively. In order to see that also the limit in (3.21) vanishes we have a closer look at the asymptotic expansion of the function $\sum_{i=1}^{n-1} R_i(\lambda)g_-(\varepsilon, \mu_i)$. Note

$$\sum_{i=1}^{n-1} \mu_i^l R_i(\lambda) = \lambda^l \quad \text{for } l = 0, 1, \ldots, n - 2.$$ 

Indeed, the difference $\sum_{i=1}^{n-1} \mu_i^l R_i(\lambda) - \lambda^l = \sum_{i=1}^{n-1} (\mu_i^l - \lambda^l) R_i(\lambda)$ is a polynomial of degree less than $n - 1$, however it has $n - 1$ zeros since it vanishes for $\mu_1, \ldots, \mu_{n-1}$.

Using the expansion (2.6) for $g_-(\varepsilon, \cdot)$ this implies

$$W\left(\sum_{i=1}^{n-1} R_i(\lambda)g_-(\varepsilon, \mu_i), g_-(\varepsilon, \lambda)\right) =$$

$$= W\left(\sum_{j=1}^{m_0} c_j(\lambda)\varepsilon^{\alpha_j + j} + K(\lambda)\varepsilon^{\alpha_0} + \varepsilon^{m_0 + \alpha_0 + 1}(h_1(\varepsilon) + \ln x h_2(\varepsilon))\right),$$

$$\sum_{j=1}^{m_0} c_j(\lambda)\varepsilon^{\alpha_j + j} + K(\lambda)\varepsilon^{\alpha_0} + \varepsilon^{m_0 + \alpha_0 + 1}(h_3(\varepsilon) + \ln x h_4(\varepsilon))\right),$$

where $h_i$ for $i = 1, \ldots, 4$ are holomorphic at $\varepsilon = 0$. Since the singular terms of the two functions here coincide the limit for $\varepsilon \to 0$ indeed is zero, and we finally obtain from (3.20)

$$d(\lambda) - m(\lambda) = b_{n-2}(\lambda)(Q_M(\lambda) + \cot \theta) - \sum_{i=1}^{n-1} R_i(\lambda)m(\mu_i),$$

which, indeed, is a polynomial of degree $\leq n - 2$. \hfill \Box

**Corollary 3.5.** The generalized Titchmarsh-Weyl-coefficient $m_i$ introduced in (2.7), belongs to the generalized Nevanlinna class $\mathcal{N}_n$ where

$$\kappa = \left[ \frac{n - 1}{2} \right] = \left[ \frac{1 + \sqrt{1 + 4\theta}}{2} \right].$$
This improves the result in [6] where \( \kappa \leq \left[ \frac{n-1}{2} \right] \) was shown.

Proof. In [3, Sections 4.4 and 5.2] it was shown that the function \( d \) admits also a minimal representation in a certain Pontryagin space and hence belongs to the class \( \mathcal{N}_\kappa \) with \( \kappa = \left[ \frac{n-1}{2} \right] \). In fact, it belongs even to the class \( \mathcal{N}_\kappa^\infty \) and hence has an irreducible representation of the form

\[
d(\lambda) = (\lambda^2 + 1)^\kappa d_0(\lambda) + p_{2\kappa-1}(\lambda),
\]

where \( d_0 \) is a usual Nevanlinna function satisfying

\[
\lim_{y \to \infty} \frac{\text{Im} \, d_0(iy)}{y} = 0, \quad \lim_{y \to \infty} y \, \text{Im} \, d_0(iy) = \infty
\]

and \( p_{2\kappa-1} \) is a polynomial of degree at most \( 2\kappa - 1 \). In order to determine the negative index of \( m \), according to Theorem 3.1, one has to add another polynomial of degree at most \( n-2 \). Since \( n-2 \leq 2\kappa - 1 \) also \( m \) admits an irreducible representation of the form (3.25). This, in particular, implies \( m \in \mathcal{N}_\kappa \). \( \square \)

References


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