The Renormalization of Periodic Jacobi Matrices and Limit Periodic Matrices with the singular continuous spectrum

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80's: discovered some of the most basic general properties of Jacobi matrices with almost periodic coefficients

- J. Avron, B. Simon, *Singular continuous spectrum for a class of almost periodic Jacobi matrices*, Bull. AMS **6** (1982), 81–85.

- J. Bellissard, B. Simon, *Cantor spectrum for almost Mathieu equation*, J. Funct. Anal., **48** (1982), no. 3, 408–419.

direct... and inverse spectral theory methods

- J. Bellissard, D. Bessis, P. Moussa, *Chaotic states of almost periodic Schrödinger operators*, Phys. Rev. Lett. **49** (1982), no. 10, 701–704.

direct...for a recent historical account see

- Y. Last, *Spectral theory of Sturm-Liouville operators on infinite intervals: a review of recent developments.* Sturm-Liouville theory, 99–120, Birkhäuser, Basel, 2005.

- A. Avila, S. Jitomirskaya, *The Ten Martini Problem*, arXiv: math.DS/0503363.

Theorem 1 *Let $T(z) = z^2 - \lambda$, $\lambda > 3$. Let μ be the balanced T-invariant probability measure*

$$\int f(y)\, d\mu(y) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{T^{\circ n}(y)=0} f(y).$$

Let P_n be orthonormal polynomials with respect to μ. Then the corresponding recurrence coefficients $\{p_n\}$,

$$zP_n(z) = p_{n+1}P_{n+1}(z) + p_n P_{n-1}(z),$$

form a limit periodic sequence. Moreover

$$|p_{k2^n+s} - p_s| \le Ac^n, A > 0, c = c(\lambda) < 1.$$

Remark. μ is supported on the Julia set of T (a Cantor set of zero Lebesgue measure).

Open problem. Is it true for $\lambda > 2$?

- M. F. Barnsley, J. S. Geronimo, A. N. Harrington, *Almost periodic Jacobi matrices associated with Julia sets for polynomials*, Comm. Math. Phys. **99** (1985), no. 3, 303–317.

- J. Herndon, *Limit periodicity of sequences defined by certain recurrence relations; and Julia sets*, Ph.D. thesis, Georgia Institute of Technology, 1985.

- O. Knill, *Isospectral deformations of random Jacobi operators*. Comm. Math. Phys. 151 (1993), no. 2, 403–426.

- M. Shapiro, V. Vinnikov, P. Yuditskii, *Finite difference operators with a finite band spectrum*. Mat. Fiz. Anal. Geom. **11** (2004), no. 3, 331–340.

Periodic Matrices: Spectrum

Theorem 2 *A system of intervals*

$$E = [-1, 1] \setminus \{\cup(a_j, b_j)\}$$

is the spectrum of a periodic Jacobi matrix if and only if there exists a polynomial U with all critical points on $(-1, 1)$ and all critical values

$$|U(c_i)| \geq 1, \quad U'(c_i) = 0,$$

such that $E = U^{-1}([-1, 1])$.

Def. U is expanding if $|U(c_i)| > 1$ and *sufficiently* expanding if $|U(c_i)| > 10$.

Periodic Matrices: Description of $J(E)$

Having the set E of the above form fixed, let

$$J(E) := \{J \text{ is periodic}, \sigma(J) = E\}.$$

Associate with U the h.–e. Riemann surface

$$X = \{Z = (z, \lambda) : \lambda^2 - 2U(z)\lambda + 1 = 0\}.$$

Note $X_+ = \{Z \in X : |\lambda(Z)| < 1\} \simeq \bar{\mathbb{C}} \setminus E$.

Real subtorus $D(E)$ of $Jac(X)$:

$$D(E) = \{D := \sum_{i=1}^{g} Z_i - D_C : \; z(Z_i) \in [a_i, b_i]\},$$

here D_C is a point of normalization

$$D_C := \sum_{i=1}^{g} C_i, \; z(C_i) = c_i, |\lambda(C_i)| > 1.$$

Evident, topologically, $D(E) \simeq \mathbb{R}^g/\mathbb{Z}^g$.

Theorem 3 *For given $E = U^{-1}([-1, 1])$ there exists one–to–one correspondence between $J(E)$ and $D(E)$.*

The Map $D(\tilde{E}) \to D(T^{-1}(\tilde{E}))$

Let

$$U = \tilde{U} \circ T.$$

Then we have a covering

$$\pi_T : X \to \tilde{X}, \quad \pi_T(z, \lambda) = (T(z), \lambda).$$

This covering generates

$$\pi_T^* : Jac(\tilde{X}) \to Jac(X).$$

For $\tilde{D} = \sum \tilde{Z} - \sum \tilde{C} \in D(\tilde{E})$, put

$$\pi_T^*(\tilde{D}) := \sum_{\pi_T(Z)=\tilde{Z}} Z - \sum_{\pi_T(C)=\tilde{C}} C.$$

The key element of the construction — The Renormalization

$$\begin{array}{ccc} D(\tilde{E}) & \xrightarrow{\pi_T^*} & D(E) \\ \downarrow & & \downarrow \\ J(\tilde{E}) & \xrightarrow{\mathcal{J}_T} & J(E) \end{array}$$

Thus we have

$$\begin{aligned} \mathcal{J}_T \ \ & : \{\tilde{J} \text{ is periodic} : \sigma(\tilde{J}) \subset [-1,1]\} \\ & \to \{J \text{ is periodic} : \sigma(J) \subset T^{-1}([-1,1])\} \end{aligned}$$

Theorem 4 *If T is sufficiently expanding then*

$$||\mathcal{J}_T(\tilde{J}_1) - \mathcal{J}_T(\tilde{J}_2)|| \le \kappa ||\tilde{J}_1 - \tilde{J}_2||.$$

with an absolute constant $\kappa < 1$ (does not depend on T also).

Lemma 5 *Let $J = \mathcal{J}_T(\tilde{J})$. This matrix satisfies the following Renormalization Equation*

$$V^*(z-J)^{-1}V = (T(z)-\tilde{J})^{-1}T'(z)/d,$$

or, equivalently,

$$V^*T(J) = \tilde{J}V^*, \quad V^*\frac{T(z)-T(J)}{z-J}V = T'(z)/d,$$

where $V|k\rangle = |dk\rangle$, $d = \deg T$.

Lemma 6 *Assume that two measures σ and $\widehat{\sigma}$ are mutually absolutely continuous. Moreover, that $d\widehat{\sigma} = f\,d\sigma$ and*

$$(1+\epsilon)^{-1} \le f \le (1+\epsilon).$$

Let us associate with these measures Jacobi matrices $J = J(\sigma)$, $\hat{J} = J(\widehat{\sigma})$. Then for their coefficients we have

$$|\widehat{p}_s - p_s| \le \epsilon ||J||, \quad s \ge 0.$$

Constructing J with s.c. spectrum.

Point out the following two properties of the function $\mathcal{J}_T(\tilde{J})$. Due to the commutant relation $VS = S^d V$ one gets

$$\mathcal{J}_T(S^{-m}\tilde{J}S^m) = S^{-dm}\mathcal{J}_T(\tilde{J})S^{dm}.$$

The second property is

$$\mathcal{J}_{T_1}(\mathcal{J}_{T_2}(\tilde{J})) = \mathcal{J}_{T_2 \circ T_1}(\tilde{J}),$$

that is, the chain rule holds.

Now, for the chosen system of polynomials $T_1, T_2 \ldots$, $\deg T_k = d_k$, with sufficiently large critical values, define

$$J_m = \mathcal{J}_{T_m \circ \ldots \circ T_2 \circ T_1}(\tilde{J}).$$

By the Main Theorem, the limit $J = \lim_{m\to\infty} J_m$ exists and does not depend on $\tilde{J}$. For $m > n$

$$||\mathcal{J}_{T_m \circ \ldots \circ T_1}(\tilde{J}_1) - \mathcal{J}_{T_n \circ \ldots \circ T_1}(\tilde{J}_2)|| \le$$

$$\kappa^n ||\mathcal{J}_{T_m \circ \ldots \circ T_{n+1}}(\tilde{J}_1) - \tilde{J}_2|| \le 2\kappa^n.$$

Moreover,

$$||J - S^{-d_1 \ldots d_l j} J S^{d_1 \ldots d_l j}|| \le ||J - S^{-j} J S^j||\, \kappa^l \le 2\kappa^l,$$

$\forall j$. This proves that J is a limit periodic matrix, in particular, it is almost periodic.

EXAMPLE: $U_n(x) = T^{\circ n}(x)$. E is the Julia set of T.

- Let μ be the balanced measure on E, — the limit of the uniform distribution:
$$\int f(x) d\mu_n(x) = \frac{1}{d^n} \sum_{U_n(x)=0} f(x),$$
Let J_+ be corresponding to this measure Jacobi matrix.

- Let ν be the Bowen–Ruelle measure, — the limit of area–proportional distribution
$$\int f(x) d\nu_n(x) = \rho^n \sum_{U_n(x)=0} \frac{f(x)}{U_n'(x)^2},$$
Let J_- be corresponding to this measure Jacobi matrix.

Theorem 7 *The block matrix*
$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} \tag{1}$$
is almost periodic Jacobi matrix, which coefficients $\mathcal{P}(n)$, $\mathcal{Q}(n)$ are continuous functions on the set of d-adic integers.

Summary: Roughly speaking, constructing a Cantor set $E \subset \ldots \subset E_{n+1} \subset E_n \ldots$, that may support the spectrum of a limit–periodic Jacobi matrix it is enough to follow this strategy: on each step the approximating set E_n should have the form of an inverse polynomial image (the spectrum of a periodic matrix).

The above statement becomes a theorem if

$$U_n = T_n \circ \ldots \circ T_2 \circ T_1.$$

$T_1, T_2 \ldots$, is a sequence of polynomials with sufficiently large critical values.

$\|E\| > 0$ *Homogeneous sets*	$\|E\| = 0$???
$\cup$	$\cup$
The standard Cantor sets of $\|E\| > 0$	*The Cantor sets with* $E_n = U_n^{-1}([-1,1])$

Recall, E is homogeneous if there exists $\eta > 0$ such that

$$(2\delta \geq)|(x-\delta, x+\delta) \cap E| \geq \eta\delta, \quad \forall x \in E, \ \forall \delta \in (0,1).$$

Theorem 8 (Sodin, Yu.) *Let E be homogeneous, then*

$$\{J \text{ is reflectionless} : \sigma(J) = E\}$$

$$= \{J \text{ is a.p.} : \sigma(J) = \sigma_{a.c.}(J) = E\}.$$

In combination with a Lipschitz transform

Let

$$\mathcal{L} \quad : \{J \text{ is periodic} : \sigma(J) \subset [-1,1]\} \\ \to \{J \text{ is periodic} : \sigma(J) \subset [-1,1]\}$$

be such that $\mathcal{L}(S^{-1}JS) = S^{-1}\mathcal{L}(J)S$ and

$$||\mathcal{L}(J_1) - \mathcal{L}(J_2)|| \leq A||J_1 - J_2||, A > 0. \quad (2)$$

Then, increasing if necessary the constant in the definition of sufficiently expanding polynomials (to make κ smaller, so that still $A\kappa < 1$), we get a new contraction $\mathcal{J}_{T,\mathcal{L}} := \mathcal{L}\mathcal{J}_T$ that we can use instead of and/or combine with the original one, extending widely the set of a.p. Jacobi matrices with the given spectrum.

$$J = \lim_{n\to\infty} \mathcal{J}_{T_1,\mathcal{L}_1}(...\mathcal{J}_{T_n,\mathcal{L}_n}(\tilde{J})).$$

Example 1. Darboux Transform. For $a > 1$ put

$$\mathcal{L}(J) + a = \Phi\Phi^*, \text{ for } J + a = \Phi^*\Phi.$$

Proof of (2). Use the Main Theorem with

$$T(z) = (a+1)z^2 - a.$$

Example 2. Involution. Let $\iota|n\rangle := |1-n\rangle$. Put

$$J^\iota := \iota^{-1}J\iota \text{ and } \mathcal{J}_T^\iota(J) := (\mathcal{J}_T(J^\iota))^\iota.$$

Expanding transform $\pi(v) = \tau v - \frac{\tau-1}{v}$**,** $\tau > 1$.

Note that $v = 1$ is the positive fixed point, $\pi(1) = 1$. Put $E_0 = [-1, 1]$.

For a continuous function f on

$$E_1 = \pi^{-1}([-1,1]) = [-1, -1 + \frac{1}{\tau}] \cup [1 - \frac{1}{\tau}, 1]$$

we define

$$(\mathcal{L}f)(x) = \frac{1}{2} \sum_{\pi(v)=x} f(v). \tag{3}$$

The conjugate operator acts on measures

$$\mathcal{L}^* : C(E_0)^* \to C(E_1)^*.$$

Let $f_0, f_1, f_2, ...$ be a certain orthonormal system with respect to a (positive) measure $\nu \in C(E_0)^*$ then

$$f_0 \circ \pi, f_1 \circ \pi, f_2 \circ \pi, ...$$

is orthonormal system with respect to $\mu := \mathcal{L}^* \nu$. Note that if the first system form basis in $L^2_{d\nu}$ the second one form basis in the set of "even" functions from $L^2_{d\mu}$, the functions that are invariant with respect to the substitution $f(-\frac{\tau-1}{\tau v}) = f(v)$.

Example 9 *Let $f_0, f_1, f_2, ...$ be orthonormal polynomials in $L^2_{d\nu}$. $f_0 \circ \pi, f_1 \circ \pi, f_2 \circ \pi, ...$ is a certain orthonormal system in $L^2_{d\mu}$ consisting of "polynomials" of v and $1/v$, similarly to the systems that generate CMV matrices:*

$$1, v, 1/v, v^2, 1/v^2...$$

Making a small modification in this procedure, we orthogonalze

$$1, \tau v + \frac{\tau - 1}{v}, \tau v - \frac{\tau - 1}{v}, (\tau v)^2 - \left(\frac{\tau - 1}{v}\right)^2, (\tau v)^2 + \left(\frac{\tau - 1}{v}\right)^2 \dots$$

and denote the orthonormal system by $e_0, e_1, e_2, ...$

It is evident that

$$e_{2k} = f_k \circ \pi$$

and

$$e_{2k+1}(v) = \left(\tau v + \frac{\tau - 1}{v}\right) g_k(\pi(v)),$$

where g_k is also orthonormal system of polynomials but with respect to the measure $(x^2 + 4\tau(\tau - 1))d\nu(x)$, since

$$\left(\tau v + \frac{\tau - 1}{v}\right)^2 = x^2 + 4\tau(\tau - 1), \quad for\ x = \tau v - \frac{\tau - 1}{v}.$$

Let J be the Jacobi matrix, corresponding to the multiplication operator in $L^2_{d\nu}$ with respect to the basis of the orthonormal polynomials.

The given J we want to describe the multiplication operator in $L^2_{d\mu}$ with respect to $\{e_k\}$.

We decompose $L^2_{d\mu}$ onto even and odd subspaces. Then

$$\tau v - \frac{\tau-1}{v} \mapsto \begin{bmatrix} J & 0 \\ 0 & J_* \end{bmatrix},$$

where J_* is the Jacobi matrix corresponding to the measure $(x^2 + 4\tau(\tau-1))d\nu(x)$.

Futher,

$$\tau v + \frac{\tau-1}{v} \mapsto \begin{bmatrix} 0 & \Phi^* \\ \Phi & 0 \end{bmatrix}.$$

It is quite evident that Φ is an upper triangular matrix.

We get that

$$v \mapsto \frac{1}{2\tau} \begin{bmatrix} J & \Phi^* \\ \Phi & J_* \end{bmatrix},$$

and

$$-1/v \mapsto \frac{1}{2(\tau-1)} \begin{bmatrix} J & -\Phi^* \\ -\Phi & J_* \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} J^2 - \Phi^*\Phi & -J\Phi^* + \Phi^* J_* \\ \Phi J - J_*\Phi & {J_*}^2 - \Phi\Phi^* \end{bmatrix} = \begin{bmatrix} -4\tau(\tau-1) & 0 \\ 0 & -4\tau(\tau-1) \end{bmatrix}.$$

Thus Φ can be found in the upper–lower triangular factorization

$$\Phi^*\Phi = J^2 + 4\tau(\tau-1), \tag{4}$$

and for J_* we have $J_* = \Phi J \Phi^{-1}$.

Thus for

$$J = \begin{bmatrix} 0 & p_1 & & \\ p_1 & 0 & p_2 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

we have

$$\pi^*(J) = \frac{1}{2\tau} \begin{bmatrix} 0 & & & & & & \\ \lambda_0 & 0 & & & & & \\ p_1 & 0 & 0 & & & & \\ 0 & p_1 \frac{\lambda_1}{\lambda_0} & \lambda_1 & 0 & & & \\ 0 & 0 & p_2 & 0 & 0 & & \\ 0 & 0 & 0 & p_2 \frac{\lambda_2}{\lambda_1} & \lambda_2 & 0 & \\ 0 & \frac{p_1 p_2}{\lambda_0} & 0 & 0 & p_3 & 0 & 0 \\ & & \ddots & & & \ddots & \ddots \end{bmatrix}.$$

The matrix is selfadjoint and λ_n are defined by the recursion

$$\lambda_n^2 = 4\tau(\tau - 1) + p_{n+1}^2 + p_n^2 - \frac{p_n^2 p_{n-1}^2}{\lambda_{n-2}^2}. \tag{5}$$

with the initial data

$$\lambda_0^2 = p_1^2 + 4\tau(\tau - 1), \quad \lambda_1^2 = p_1^2 + p_2^2 + 4\tau(\tau - 1). \tag{6}$$

Definition. Let A be a self–adjoint operator acting in l^2_+ with a cyclic vector $|0\rangle$ and the spectrum on $[-1,1]$. We define its transform $\pi^*(A)$ in the following steps. First we define the upper triangular matrix Φ (with positive diagonal entries) by the condition

$$A^2 + 4\tau(\tau - 1) = \Phi^*\Phi. \tag{7}$$

Introduce $A_* := \Phi A \Phi^{-1}$ and define the operator

$$\begin{bmatrix} A & \Phi^* \\ \Phi & A_* \end{bmatrix}, \tag{8}$$

acting in $l^2_+ \oplus l^2_+$. Finally, using the unitary operator $U : l^2_+ \to l^2_+ \oplus l^2_+$, such that

$$U|2k\rangle = |k\rangle \oplus 0, \quad U|2k+1\rangle = 0 \oplus |k\rangle \tag{9}$$

we construct

$$\pi^*(A) := \frac{1}{2\tau} U^* \begin{bmatrix} A & \Phi^* \\ \Phi & A_* \end{bmatrix} U : l^2_+ \to l^2_+. \tag{10}$$

Theorem 10 *Let ν be the spectral measure of A,*

$$\int \frac{d\nu(x)}{x-z} = \langle 0|(A-z)^{-1}|0\rangle. \qquad (11)$$

Then $\pi^*(A)$ is a self–adjoint operator with the cyclic vector $|0\rangle$ and the spectral measure $\mu = \mathcal{L}^*\nu$.

Theorem 11 *The iterative procedure*

$$A_{n+1} = \pi^*(A_n)$$

converges to the operator $A = \lim_{n\to\infty} A_n$ with the spectral measure μ which is the eigen–measure for the Ruelle operator $\mathcal{L}^*\mu = \mu$. The operator A is the multiplication operator by independent variable in $L^2_{d\mu}$ with respect to the following basis

$$e_{2k}(v) = e_k(\pi(v)),$$

$$e_{2k+1}(v) = \left(\tau v + \frac{\tau-1}{v}\right)\sum_{j=0}^{k} c_j^k e_j(\pi(v)), \quad e_0(v) = 1,$$

where the coefficients c_j^k with $c_k^k > 0$ are uniquely determined due to the orthogonality condition $\langle e_{2k+1}, e_l\rangle = \delta_{2k+1,l}$, $l \le 2k+1$. Moreover, $e_k(v)$ is a rational function of v such that $e_k(A)|0\rangle = |k\rangle$, and

$$\begin{bmatrix} c_0^0 & c_0^1 & c_0^2 & \cdots \\ 0 & c_1^1 & c_1^2 & \cdots \\ & \ddots & \ddots & \ddots \end{bmatrix} = \Phi^{-1}. \qquad (12)$$