The Ornstein-Uhlenbeck semigroup in exterior Lipschitz domains

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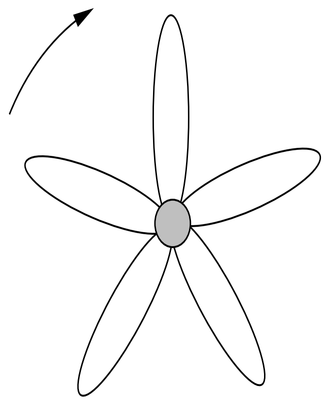
Motivation

Flow around a rotating body $\Omega(t) = e^{-tM}\Omega$



$$\begin{cases} u' - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \end{cases}$$

+ boundary & initial conditions

Transformation of coordinates $x = e^{-tM}y$

$$\begin{cases} u' - \Delta u + u \cdot \nabla u - Mx \cdot \nabla u + Mu + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \end{cases}$$

Consider operator $\Delta + Mx \cdot \nabla$

C_0-semigroups

$$u'(t) = Au(t) \quad \text{for } t > 0, \quad u(0) = u_0$$

- A generator, $u_0 \in D(A) \Rightarrow$ unique solution: $u(t) = T(t)u_0$
- A dissipative in $L^p(\Omega)$ iff for every $u \in D(A) \subseteq L^p(\Omega)$,

$$\operatorname{Re} \langle Au, u^* \rangle \leq 0 \text{ where } u^* = |u|^{p-2}\overline{u}.$$

- Lumer, Phillips: A densely defined, dissipative. TFAE
  1. $\overline{A}$ generates a contractive C_0-semigroup.
  2. $\operatorname{rg}(\lambda - A)$ dense for one/all $\lambda > 0$.
- T is analytic if it has an analytic extension to a sector Σ_θ, bounded on $\Sigma_{\theta'} \cap \{z \in \mathbb{C} : |z| \leq 1\}$, $0 < \theta' < \theta$.

References

Probability	Kolmogorov ’34	representation on $\mathbb{R}^n$
	Meyer ’82	generation in $L^p_\mu(\mathbb{R}^n)$
Invariant measure	Goldys ’99	analyticity in $L^2_\mu(\mathbb{R}^n)$
	Metafune et al ’02	analyticity and domain in $L^p_\mu(\mathbb{R}^n)$
Lebesgue measure	Lunardi et al ’95, ’97	generation in $BUC(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$
	Hishida ’99, ’00	generation in $L^2(\Omega)$, $L^2_\sigma(\Omega)$, Ω smooth
	Metafune ’01	spectrum in $L^p(\mathbb{R}^n)$
	Metafune et al ’02	domain in $L^p(\mathbb{R}^n)$
	Prüss et al ’05	superlinear drift in $L^p(\mathbb{R}^n)$
	Hieber et al ’05	exterior domains, L^p-L^q estimates
Nonlinear	Hishida ’99	local uniqueness in $L^2_\sigma(\Omega)$, Ω smooth, "nice" initial data
	Hieber et al ’05	local uniqueness in $L^p_\sigma(\mathbb{R}^n)$, $p \geq n$, and in $L^p_\sigma(\Omega)$, Ω smooth, $p \geq n$

Laplacian in Lipschitz domains

- $D(\Delta_p^w) = \{u \in W_0^{1,p}(D) : \Delta u \in L^p(D)\}, \Delta_p^w u = \Delta u$
- D bounded convex, $1 < p \leq 2 \Rightarrow$
  $D(\Delta_p^w) = W^{2,p}(D) \cap W_0^{1,p}(D)$

Theorem 1

1. $D \subseteq \mathbb{R}^n$, $n \geq 3$, *bounded Lipschitz domain*
   $\varepsilon > 0$ *depending only on the Lipschitz constant of* D,
   $(3+\varepsilon)' < p < 3+\varepsilon$
2. *if* $n = 2$, $(4+\varepsilon)' < p < 4+\varepsilon$
3. D *bounded convex domain in* $\mathbb{R}^n$, $n \geq 2$, $1 < p < \infty$

Δ_p^w *generates an analytic* C_0*-semigroup of contractions on* $L^p(D)$.

Drift Operator

$$\begin{cases} B_D u(x) := Mx \cdot \nabla u(x), \; x \in D, \\ D(B_D) := \{u \in W^{1,p}(D) : Mx \cdot \nabla u \in L^p(D)\}. \end{cases}$$

Lemma 1

- $D \subseteq \mathbb{R}^n$ *Lipschitz domain,* $1 < p < \infty$

Then $B_D + \frac{\mathrm{tr}\ M}{p}$ *is dissipative in* $L^p(D)$*.*

Lemma 2

- D *bounded Lipschitz domain in* $\mathbb{R}^n$
- $(3+\varepsilon)' < p < 3+\varepsilon$

B_D *is relatively bounded by* Δ_p^w *in* $L^p(D)$*. The relative bound is given by zero.*

$\mathbb{R}^n$-semigroup

$$\begin{cases} A_{\mathbb{R}^n} u(x) := \Delta u(x) + Mx \cdot \nabla u(x), \; x \in \mathbb{R}^n, \\ D(A_{\mathbb{R}^n}) := \{u \in W^{2,p}(\mathbb{R}^n) : Mx \cdot \nabla u \in L^p(\mathbb{R}^n)\} \end{cases}$$

Theorem 2 $1 < p < \infty$, $A_{\mathbb{R}^n}$ *generates positive* C_0*-semigroup on* $L^p(\mathbb{R}^n)$ *with representation*

$$e^{tA_{\mathbb{R}^n}} f(x) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} f(e^{tM}x - y) e^{-\frac{1}{4}(Q_t^{-1}y,y)} \mathrm{d}y,$$

for $x \in \mathbb{R}^n, \; t > 0$ *with* $Q_t := \int_0^t e^{sM} e^{sM^T} \, \mathrm{d}s$. *For* $p < q$,

- $\left\| e^{tA_{\mathbb{R}^n}} f \right\|_q \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{\omega t} \|f\|_p \, , \; t > 0, \; f \in L^p(\mathbb{R}^n)$
- $\left\| \nabla e^{tA_{\mathbb{R}^n}} f \right\|_q \leq C t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{\omega t} \|f\|_p \, , \; t > 0, \; f \in L^p(\mathbb{R}^n)$

$\mathbb{R}^n$-resolvent

Proposition 1 *There exists $\lambda_0 \in \mathbb{R}$ such that for $\lambda > \lambda_0$, the unique solution of the resolvent problem*

$$(\lambda - A_{\mathbb{R}^n})u = f$$

satisfies

- $\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Mx \cdot \nabla u\|_{L^p(\mathbb{R}^n)} \leq C_\lambda \|f\|_{L^p(\mathbb{R}^n)}$
- $\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq \frac{C}{|\lambda - \omega|^{\frac{1}{2}}} \|f\|_{L^p(\mathbb{R}^n)}$

for some constants C, ω.

Bounded domains

$$\begin{cases} A_D u(x) := \Delta u(x) + Mx \cdot \nabla u(x), \ x \in D, \\ D(A_D) := \{u \in W_0^{1,p}(D) : \Delta u \in L^p(D)\}. \end{cases}$$

Proposition 2

- *D bounded Lipschitz domain in $\mathbb{R}^n$, $(3+\varepsilon)' < p < 3+\varepsilon$*

A_D generates analytic quasi-contractive semigroup T on $L^p(D)$ with $\omega(T) \leq -\mathrm{tr}\ M/p$.
Solutions of $(\lambda - A_D)u = f$ satisfy

- $\|u\|_{L^p(D)} + \|\Delta u\|_{L^p(D)} + \|Mx \cdot \nabla u\|_{L^p(D)} \leq C_\lambda \|f\|_{L^p(D)}$ *for* $\mathrm{Re}\lambda > -\mathrm{tr}\ M/p$
- $\|u\|_{L^p(D)} \leq \frac{C}{|\lambda + \frac{\mathrm{tr}\ M}{p}|} \|f\|_{L^p(D)}$ *for* $\lambda \in -\frac{\mathrm{tr}\ M}{p} + \Sigma_\varphi$
- $\|\nabla u\|_{L^p(D)} \leq \frac{C}{\lambda^\Theta} \|f\|_{L^p(D)}$, $\theta > 0$ *for* $\lambda > \max\left\{-\frac{\mathrm{tr}\ M}{p}, 0\right\}$

Exterior domains

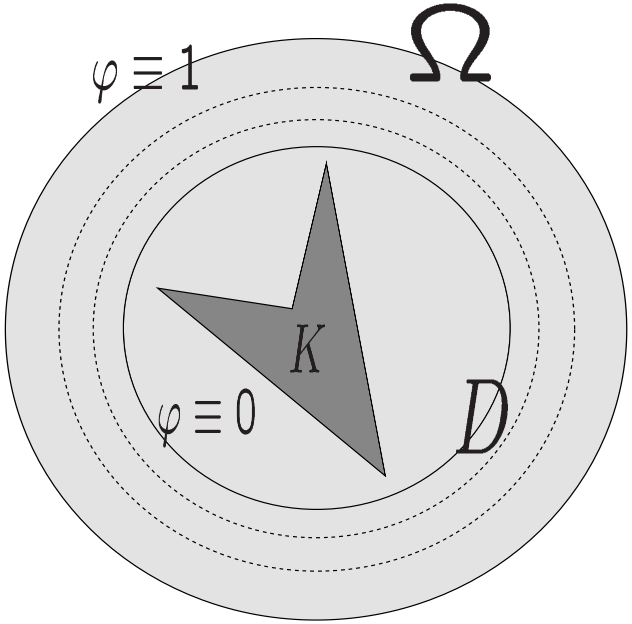
$$\begin{cases} A_\Omega u(x) := \Delta u(x) + Mx \cdot \nabla u(x), \; x \in \Omega, \\ D(A_\Omega) := \{u \in W_0^{1,p}(\Omega) : \Delta u \in L^p(\Omega) \text{ and } Mx \cdot \nabla u \in L^p(\Omega)\}. \end{cases}$$

Theorem 3 *Ω exterior Lipschitz domain, $(3+\varepsilon)' < p < 3+\varepsilon$. A_Ω generates a quasi-contractive C_0-semigroup T on $L^p(\Omega)$ with $\omega(T) \leq -\frac{\mathrm{tr}\ M}{p}$.*

Proof:

- $B_\Omega + \frac{\mathrm{tr}\ M}{p}$ is dissipative
- $p \geq 2 \Rightarrow \Delta_p^w$ is dissipative
- Lumer-Phillips $\Rightarrow$ result for $2 \leq p < 3 + \varepsilon$ if range condition satisfied
- for $(3+\varepsilon)' < p < 2$, use duality argument for dissipativity

$(\lambda - A_\Omega)u = f$

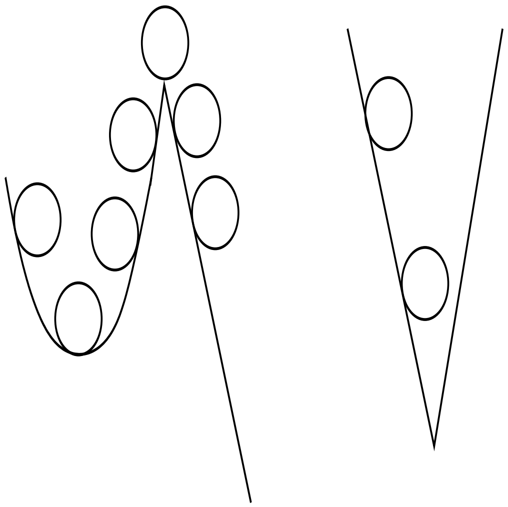


- $(\lambda - A_{\mathbb{R}^n})u_0 = f_0$ in $\mathbb{R}^n$
  $\Rightarrow u_0 \in W^{2,p}(\mathbb{R}^n), Mx \cdot \nabla u_0 \in L^p(\mathbb{R}^n)$
- $(\lambda - A_D)u_* = f_*$ in D
  $\Rightarrow u_* \in W_0^{1,p}(D), \Delta u_* \in L^p(D)$
- Ansatz: $u = \varphi u_0 + (1-\varphi)u_*$

- $u \in W_0^{1,p}(\Omega)$, $\Delta u \in L^p(\Omega)$, $Mx \cdot \nabla u \in L^p(\Omega)$
- $(\lambda - A_\Omega)u = (I - T_\lambda)f$
- $T_\lambda f := 2\nabla\varphi \cdot (\nabla u_0 - \nabla u_*) + \Delta\varphi(u_0 - u_*) + (Mx \cdot \nabla\varphi)(u_0 - u_*)$

For large $\lambda > 0$, $I - T_\lambda$ is invertible.

Uniform outer ball condition



- Ω exterior Lipschitz domain, uniform outer ball condition
- $$\begin{cases} A_\Omega u(x) = \Delta u(x) + Mx \cdot \nabla u(x), \ x \in \Omega, \\ D(A_\Omega) = \{u \in W^{2,p}(\Omega) : Mx \cdot \nabla u \in L^p(\Omega)\} \\ \qquad\qquad \cap\, W_0^{1,p}(\Omega). \end{cases}$$

Theorem 4 *For $1 < p \le 2$, A_Ω generates quasi-contractive C_0-semigroup T on $L^p(\Omega)$ with $\omega(T) \le -\mathrm{tr}\, M/p$ and for $p < q \le 2$*

- $$\left\| e^{tA_\Omega} f \right\|_q \le C t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\widetilde{\omega} t} \left\| f \right\|_p, \ t > 0$$
- $$\left\| \nabla e^{tA_\Omega} f \right\|_q \le C t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\widetilde{\omega} t} \left\| f \right\|_p, \ t > 0$$