PARSEVAL FORMULA FOR WAVE EQUATIONS WITH DISSIPATIVE TERM OF RANK ONE

MITSUTERU KADOWAKI (EHIME UNIV.),
HIDEO NAKAZAWA (CHIBA INST. TECH.),
KAZUO WATANABE (GAKUSHUIN UNIV.)

1. Introduction and Problems

Consider the wave equation with a dissipative term in 1-dimension:

$$\partial_t^2 u(x,t) + \langle \partial_t u, \varphi \rangle_0 \varphi(x) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{1}$$

where $\langle \cdot, \cdot \rangle_0$ is the usual inner product on $L^2(\mathbb{R})$, and $\varphi \in L^2_s(\mathbb{R})$ $(s \geq 1/2)$.

We deal with (1) as a perturbed system of

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{2}$$

Put $f(t) = {}^t(u(x,t), \partial_t u(x,t))$. Then (1) and (2) can be written as

$$\partial_t f(t) = -iAf(t) \qquad \text{and} \quad \partial_t f(t) = -iA_0 f(t),$$

where

$$A = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 & -\langle \cdot, \varphi \rangle_0 \varphi \end{pmatrix} \qquad \text{and} \quad A_0 = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix},$$

respectively.

$\mathcal{H}$: Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} (\partial_x f_1(x) \overline{\partial_x g_1(x)} + f_2(x) \overline{g_2(x)}) dx,$$

where $f = {}^t(f_1, f_2)$ and $g = {}^t(g_1, g_2)$. The norm of $\mathcal{H}$ is denoted by $\| \cdot \|$.

Then we know that the domain of A_0, A is

$$D(A) = D(A_0) = \{ f = {}^t(f_1, f_2) \in \mathcal{H}; \partial_x^2 f_1 \in L^2(\mathbb{R}), f_2 \in H^1(\mathbb{R}) \},$$

where $H^s(\mathbb{R})$ is Sobolev space.

We use the following notation:

$$R(z) = (A - z)^{-1} \ (z \in \rho(A)), \quad R_0(z) = (A_0 - z)^{-1} \ (z \in \rho(A_0)),$$
$$r_0(z) = (-\partial^2/\partial x^2 - z)^{-1} \ (z \in \mathbb{C} \setminus [0, \infty)).$$

Theorem 1. *(*Mochizuki,'76*)*

1. *A has no real eigenvalues.*
2. *$W = s - \lim_{t \to \infty} e^{itA_0} e^{-itA}$ exists as a* **non-trivial** *operator from $\mathcal{H}$ to $\mathcal{H}$.*

$$f(t) = e^{-itA} f, f(0) = f \in \mathcal{H} \tag{3}$$

Theorem 1 (1) implies that (3) does not have the bound mode which is also

$$\lim_{t \to \infty} \| e^{-itA} f \| \neq 0$$

for some $f \in \mathcal{H}$, but not scattering mode.

(Q1) Does (3) have the dissipative mode ? This mode means

$$\lim_{t\to\infty} \|e^{-itA}f\| = 0$$

for some $f(\neq 0) \in \mathcal{H}$.

(Q2) Is it true that (3) is equal to the linear combination of modes which are found ?

Our aim is to answer (Q1) and (Q2).

Assumptions:

(A1)
$$\varphi(x) \in L^2_{1+s}(\mathbb{R})$$

for some $s > 1/2$ and

(A2)
$$\Phi(\lambda) \leqq \Phi(\mu) \quad (0 \leqq \mu \leqq \lambda),$$

where

$$\Phi(\lambda) = |\hat{\varphi}(\lambda)|^2 + |\hat{\varphi}(-\lambda)|^2.$$

For $\lambda < 0$ dealing with $\Phi(-\lambda)$ we extend $\Phi(\lambda)$ as a function on $\mathbb{R}$.

First, we give the spectrum of the operator A, secondly an result for (generalized) Parseval's formula for A.

We put

$$\Sigma_\pm = \{z \in \mathbb{C}_\pm; \mathit{\Gamma}(z) = 0\} \quad \text{and} \quad \Sigma^0_\pm = \{\lambda \in \mathbb{R}; \mathit{\Gamma}(\lambda \pm i0) = 0\},$$

where

$$\mathit{\Gamma}(z) = 1 - iz\langle r_0(z)\varphi, \varphi\rangle_0, \quad \mathit{\Gamma}(\lambda \pm i0) = 1 - i\lambda\langle r_0(\lambda \pm i0)\varphi, \varphi\rangle_0$$

and we also use $\langle\cdot,\cdot\rangle_0$ as the dual coupling of $L^2_s(\mathbb{R})$ and $L^2_{-s}(\mathbb{R})$ for some $s > 1/2$.

Noting that for $z \in \mathbb{C} \setminus \mathbb{R}$ and $f = {}^t(f_1, f_2) \in \mathcal{H}$,

(4)
$$R_0(z)f = {}^t(r_0(z)(zf_1 + if_2), i\partial_x r_0(z)\partial_x f_1 + zr_0(z)f_2)$$

Remark. $\Sigma_+ = \Sigma^0_+ = \emptyset$.

Related result: Gilliam and Shulenberger ('86) for Maxwell with the boundary condition.

2. Main theorems

Proposition 1. *Assume that (A1)and (A2). Then*

(5)
$$\Sigma_- = \begin{cases} \emptyset, & (\mathit{\Gamma}(-i0) \geqq 0), \\ \{i\kappa_0\}, & (\mathit{\Gamma}(-i0) < 0) \end{cases}$$

for some $\kappa_0 < 0$ *and*

(6)
$$\Sigma^0_- = \begin{cases} \emptyset, & (\mathit{\Gamma}(-i0) \neq 0), \\ \{0\}, & (\mathit{\Gamma}(-i0) = 0). \end{cases}$$

Moreover, for the case $\mathit{\Gamma}(-i0) < 0$ *and* $\mathit{\Gamma}(-i0) = 0$*, it holds* $\mathit{\Gamma}'(i\kappa_0) \neq 0$ *and* $\mathit{\Gamma}'(-i0) \neq 0$*,respectively.*

Remark. $\mathit{\Gamma}'(-i0) \neq 0$ means $0 \in \Sigma^0_-$ is of order 1 of zeros of $\mathit{\Gamma}(-i\lambda)$ $(\lambda \in \mathbb{R})$.

Describing main theorem(Parseval's formula corresponding to A) we need some notations.

Using the second resolvent equation and by the rank one perturbation we know $z \notin \Sigma_+$ (resp. Σ_-) if and only if $z \in \rho(A) \cap \mathbb{C}_+$ (resp. $z \in \rho(A) \cap \mathbb{C}_-$) and

(7)
$$R(z)f = R_0(z)f + \frac{i\langle f, v(\overline{z})\rangle}{\mathit{\Gamma}(z)} v(z)$$

for any $f =^t (f_1, f_2) \in \mathcal{H}$, where

$$v(z) = \begin{pmatrix} ir_0(z)\varphi \\ zr_0(z)\varphi \end{pmatrix}.$$

$$\mathcal{H}_s = \{f =^t (f_1, f_2); \int_{\mathbb{R}} (1 + |x|^2)^s (|\partial_x f_1(x)|^2 + |f_2(x)|^2) dx < \infty\}.$$

Define operator $\mathfrak{F}_0$ and $\mathfrak{F}, \mathfrak{G}$ as follows :

$$(\mathfrak{F}_0 f)(\lambda) = \begin{cases} {}^t\left(\dfrac{\lambda \hat{f}_1(\lambda) + i\hat{f}_2(\lambda)}{\sqrt{2}}, \dfrac{\lambda \hat{f}_1(-\lambda) + i\hat{f}_2(-\lambda)}{\sqrt{2}}\right), & (\lambda > 0) \\ {}^t\left(\dfrac{-\lambda \hat{f}_1(-\lambda) - i\hat{f}_2(-\lambda)}{\sqrt{2}}, \dfrac{-\lambda \hat{f}_1(\lambda) - i\hat{f}_2(\lambda)}{\sqrt{2}}\right), & (\lambda < 0), \end{cases}$$

and

$$\begin{cases} (\mathfrak{F}f)(\lambda) = (\mathfrak{F}_0 f)(\lambda) + \dfrac{i\langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)} (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda), \\ (\mathfrak{G}f)(\lambda) = (\mathfrak{F}_0 f)(\lambda) - \dfrac{i\langle f, v(\lambda - i0)\rangle}{\overline{\Gamma(\lambda - i0)}} (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda). \end{cases}$$

Remark. This operator has the following property:

$$\int_{-\infty}^{\infty} \langle (\mathfrak{F}Af)(\lambda), \tilde{g}(\lambda)\rangle_{\mathbb{C}^2} d\lambda = \int_{-\infty}^{\infty} \lambda \langle (\mathfrak{F}f)(\lambda), \tilde{g}(\lambda)\rangle_{\mathbb{C}^2} d\lambda. \tag{8}$$

for any $f \in D(A)$ and $\tilde{g} \in L^2(\mathbb{R}; \mathbb{C}^2)$.

Theorem 2. *(Parseval formula) Assume that (A1)and (A2). Then*

1. $\Gamma(-i0) \neq 0$, *it holds that*

$$\langle f, g\rangle = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2} d\lambda, & (\Gamma(-i0) > 0) \\ \displaystyle\int_{-\infty}^{\infty} \langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2} d\lambda + \langle Pf, g\rangle, & (\Gamma(-i0) < 0) \end{cases}$$

for any $f, g \in \mathcal{H}$, *where* P *is the projection for the eigenvalue* $i\kappa_0$ *of* A.

2. $\Gamma(-i0) = 0$, *it holds that for any* $f \in \mathcal{H}, g \in \mathcal{E} = \{g \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}); \langle v(-i0), g\rangle = 0\}$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2} d\lambda$$

Now we can answer (Q1) and (Q2).
By Theorem, we see that

$$\mathrm{Ker} W = \begin{cases} \{0\} & (\Gamma(-i0) \geq 0), \\ \mathrm{Range} P & (\Gamma(-i0) < 0). \end{cases} \tag{9}$$

If $\Gamma(-i0) \geqq 0$, then (3) has scattering mode only, i.e. $f \neq 0$ if and only if $Wf \neq 0$ and

$$\lim_{t \to \infty} \|e^{-itA} f - e^{-itA_0} W f\| = 0.$$

If $\Gamma(-i0) < 0$, (3) has linear combination of scattering and dissipative modes, i.e. for the decomposition

$$e^{-itA} f = e^{-itA}(f - Pf) + e^{-itA} Pf,$$

$f - Pf \neq 0$ if and only if $Wf \neq 0$ and

$$\lim_{t \to \infty} \|e^{-itA} f - e^{-itA_0} W f\| = \lim_{t \to \infty} \|e^{-itA}(f - Pf) - e^{-itA_0} W f\| = 0.$$

3. Outline of the proof of Theorem 2.

Step 1. Rewrite the time-depndent representation of the wave operator W as follows: (Kato '66)

$$\langle Wf, g\rangle = \lim_{\kappa\downarrow 0}\frac{\kappa}{\pi}\int_{-\infty}^{\infty}\langle R(\lambda+i\kappa)f, R_0(\lambda+i\kappa)g\rangle d\lambda.$$

Using (7) (the representation of the resolvent of A), $\mathfrak{F}_0$ and $\mathfrak{F}$ we know that

$$\langle Wf, g\rangle = \int_{-\infty}^{\infty}\langle(\mathfrak{F}f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} d\lambda.$$

Step 2. We can prove the following formula: for $s > 1/2$, for $f, g \in \mathcal{H}_s$ and $\lambda \notin \Sigma^0_-$

$$\begin{aligned}&\langle(\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2}\\ =&\langle(\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} + \frac{1}{2\pi}\frac{\langle f, v(\lambda-i0)\rangle\langle v(\lambda+i0), g\rangle}{\Gamma(\lambda+i0)}\\ &-\frac{1}{2\pi}\frac{\langle f, v(\lambda+i0)\rangle\langle v(\lambda-i0), g\rangle}{\Gamma(\lambda-i0)}.\end{aligned} \tag{10}$$

Hence, to prove Parseval formula we have to characterize some zeros of $\Gamma(z)$. These points are non-real eigenvaule $i\kappa_0$ as in (5) and "spectral singularity" for A.

In fact we can prove

$$\inf_{\mathrm{Im} z\geqq 0}\mathrm{Re}\Gamma(z) \geqq 1 \tag{11}$$

Because, for $z = \lambda + i\kappa \in \mathbb{C}_+$ we have

$$\Gamma(z) = 1 + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\kappa}{(r-\lambda)^2+\kappa^2}\Phi(r)dr - \frac{i}{2}\int_{-\infty}^{\infty}\frac{r-\lambda}{(r-\lambda)^2+\kappa^2}\Phi(r)dr.$$

There exists a positive constant C such that

$$\liminf_{|z|\to\infty, \mathrm{Im} z\leqq 0}|\mathrm{Re}\Gamma(z)| \geqq C. \tag{12}$$

Because we know that $z = \lambda + i\kappa \in \mathbb{C}_-$ for $\lambda, |\lambda| > 1$,

$$\lim_{\kappa\uparrow 0}\mathrm{Re}\Gamma(z) = 1 - \frac{\pi}{2}\Phi(\lambda)$$

Hence we see that the spectral singularity of A is stiuated in the bounded region of $\mathbb{C}_-$.

And using Assumption (A2) (the monotonicity of $\Psi(\lambda)$) we notice that the order of the zeros of $\Gamma(z)$ is one.

References

[1] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Math. Soc. Lect. Note Ser. No. 271, Cambridge Univ. Press, 2000

[2] M. Kadowaki *Resolvent estimates and scattering states for dissipative systems* , Publ. RIMS Kyoto Univ., **38** 2002, pp.191–209

[3] M. Kadowaki, H. Nakazawa and K. Watanabe, *On the asymptotics of solutions for some Schrödinger equations with dissipative perturbations of rank one*, Hiroshima Math.J., **34**, 2004, pp.345–369

[4] T. Kato, *Wave operators and similarity for some non–self adjoint operators*, Math. Ann. **162**, 1966, pp. 258–279

[5] S. T. Kuroda, *Spectral theory II*, Iwanami, Tokyo 1978(in Japanese)

[6] ———, *An Introduction to Scattering Theory*, Lecture Note Series N^o 51, Matematisk Institut, Aarhus University, 1980

[7] K. Mochizuki, *Scattering theory for wave equations with dissipative terms*, Publ. RIMS Kyoto Univ., **12**, 1976, pp. 383–390

[8] ———, *Inverse scattering for a small nonselfadjoint perturbation of the wave equation*, Anaiysis and application ISAAC 2001(Berlin) pp 303-316

[9] M.A. Naimark , *Investigation of the spectrum and the expansion in eigenfunction of a non-selfadjoint operator of second order on a semi axis*, Amer. Math. Soc. Transl. Ser, **216**, 1960, pp. 103–193

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, 1975, New York, San Francisco, London

[11] ———, *Methods of Modern Mathematical Physics, IV, Analysis of Operators*, Academic Press, 1978, New York, San Francisco, London

[12] J.T. Schwartz, *Some non selfadjoint operators* , Comm. Pure Appl. Math., **13**, 1960, pp. 609–639

[13] B. Simon *Phase space analysis of simple scattering systems: extensions of some work of Enss*, Duke Math. J, **46**, 1979, pp.119–168

[14] S.A. Stepin, *Scattering and spectral singularities for some dissipative operators of mathematical phisics*, J. of Nonliear Math. Phys., **11**, 2004, pp. 194–203