Trace Formulas for Jacobi Operators in Connection with Scattering Theory for Quasi-Periodic Background

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OTAMP, June 2006

Jacobi operators

Our topic is connected with scattering theory of the pair (H, H_p), where H_p is a periodic Jacobi operator,

$$H_p = a_p S^+ + S^- a_p + b_p,$$

and

$$H = aS^+ + S^- a + b$$

is a short-range perturbation (Marchenko class) of H_p.
Here $S^\pm f(n) = f(n \pm 1)$ are the usual shift operators.

Trace formulas

In the case of constant background

$$H_0 = \frac{1}{2}\left(S^+ + S^-\right)$$

trace formulas (aka Case type sum rules) play an important role both for (inverse) spectral theory and for application to completely integrable lattices (conserved quantities).
Have attracted considerable interest (e.g.): Case 75, Deift+Killip 99, Killip+Simon 03, Laptev+Naboko+Safronov 03, Nazarov+Peherstorfer+Volberg+Yuditskii 05

Question: What are the analogous of the usual trace formulas for the case of a periodic background?

Trace formulas for constant background

Classical Case type sum rules:

$$\ln\Big(\prod_{m=-\infty}^{\infty} 2a(m)\Big) = -\sum_{j=1}^{N} \ln|k_j| + \frac{1}{\pi}\int_0^{\pi} \ln|T(\mathrm{e}^{\mathrm{i}\varphi})| d\varphi$$

$$-\frac{1}{n}\mathrm{tr}\Big(T_n(H) - T_n(H_0)\Big) = -\frac{1}{\pi}\int_0^{\pi} \ln|T(\mathrm{e}^{\mathrm{i}\varphi})| \cos(n\varphi) d\varphi + \sum_{j=1}^{N} \frac{k_j^n - k_j^{-n}}{n},$$

where $T(k)$ is the transmission coefficient, $\rho_j = \frac{k_j + k_j^{-1}}{2}$ are the eigenvalues and $T_n(x)$ are the usual Chebyshev polynomials.

Proof for constant background

Three steps:

1. Identify the transmission coefficient $T(z)$ as perturbation determinant in the sense of Krein.
2. Write $T(z)$ in terms of its boundary values with the help of the Poisson-Jensen formula.
3. Expand both formulas around ∞ and compare coefficients.

Krein's spectral shift theory

The transmission coefficient $T(z)$ satisfies

$$\frac{T(\infty)}{T(z)} = \det\left(\mathbb{I} + (H - H_0)(H_0 - z)^{-1}\right)$$

and hence

$$\ln T(z) = \ln A + \sum_{n=1}^{\infty} \frac{\tau_n}{n\, z^n},$$

where

$$A = \prod_{m=-\infty}^{\infty} \frac{a(m)}{1/2}, \qquad \tau_n = \mathrm{tr}(H^n - H_0^n)$$

Poisson-Jensen formula

The transmission coefficient can be reconstructed from its boundary values:

$$T(k) = \Big(\prod_{j=1}^{N} \frac{|k_j|(k - k_j^{-1})}{k - k_j} \Big) \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{|\kappa|=1} \ln|T(\kappa)| \frac{\kappa + k}{\kappa - k} \frac{d\kappa}{\kappa} \right),$$

where $z = \frac{k+k^{-1}}{2}$.
Expanding around $k = 0$ and comparing coefficients gives the trace formulas.

Krein's spectral shift theory

The transmission coefficient $T(z)$ satisfies

$$\frac{T(\infty)}{T(z)} = \det\left(\mathbb{I} + (H - H_p)(H_p - z)^{-1}\right)$$

and hence

$$\ln T(z) = \ln A + \sum_{n=1}^{\infty} \frac{\tau_n}{n\, z^n},$$

where

$$A = \prod_{m=-\infty}^{\infty} \frac{a(m)}{a_p(m)}, \qquad \tau_n = \mathrm{tr}(H^n - H_p^n)$$

The Riemann surface

Since the spectrum of H_p consists of $g+1$ band's, the associated Riemann surface

$$y^2 = \prod_{j=0}^{2g+1} (z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1},$$

is no longer simply connected and we cannot map it to the Riemann sphere.
The spectrum of H_p now corresponds to the boundary of the *upper sheet* Π_+ and in order to reconstruct the transmission coefficient we need the Green's function and the harmonic measure of Π_+.

The Green's function and the harmonic measure

Lemma

The Green function of Π_+ with pole at p_0 is given by

$$g(z, z_0) = -\mathrm{Re} \int_{E_0}^{p} \omega_{p_0 \tilde{p}_0}, \quad p = (z, +),\ p_0 = (z_0, +),$$

where $\tilde{p}_0 = \overline{p_0}^*$ (i.e., the complex conjugate on the other sheet) and ω_{pq} is the normailzed Abelian differential of the third kind with poles at p and q.

As a consequence:

Lemma

The harmonic measure of $\partial\Pi_+$ with pole at λ is given by

$$\mu(p, \lambda) d\lambda = \frac{1}{\pi} \mathrm{Im}\, \omega_{p E_0}(\lambda).$$

Poisson-Jensen formula

Theorem

The transmission coefficient can be reconstructed from its boundary values via

$$T(z) = \left(\prod_{j=1}^{N} \exp\left(\int_{E(\rho_j)}^{\rho_j} \omega_{pp^*} \right) \right) \exp\left(\frac{1}{\pi \mathrm{i}} \int_{\Sigma} \ln |T| \omega_{pp^*} \right),$$

where $\Sigma = \sigma(H_p)$ and ρ_j are the eigenvalues of H.

Note that neither the Blaschke product nor the outer function are single-valued on Π_+ in general.

Single-valuedness

In inverse scattering theory one uses this formula to reconstruct T from the reflection coefficient R_+ (or R_-). Hence we are naturally interested in when T is single-valued for given $R_\pm$.

Theorem

The transmission coefficient T defined via the Poisson-Jensen-type formula is single-valued if and only if the eigenvalues ρ_j and the reflection coefficient $R_\pm$ satisfy

$$2\sum_{j=1}^{N}\int_{E_0}^{\rho_j}\zeta_\ell - \frac{1}{\pi \mathrm{i}}\int_\Sigma \ln(1-|R_\pm|^2)\zeta_\ell \in \mathbb{Z},$$

where ζ_ℓ is the canonical basis of holomorphic differentials.

Trace formulas

Theorem

The following trace formulas are valid:

$$\ln(A) = -\sum_{j=1}^{N} \int_{E(\rho_j)}^{\rho_j} \omega_{\infty_+\infty_-} + \frac{1}{\pi \mathrm{i}} \int_{\Sigma} \ln|T|\, \omega_{\infty_+\infty_-},$$

$$\frac{1}{n}\tau_n = -\sum_{j=1}^{N} \int_{E(\rho_j)}^{\rho_j} \omega_{n-1} + \frac{1}{\pi \mathrm{i}} \int_{\Sigma} \ln|T|\, \omega_{n-1},$$

where

$$\omega_k = \omega_{\infty_+;k-1} - \omega_{\infty_-;k-1}$$

and $\omega_{p;k}$ is the normalized Abelian differential of the second kind with a pole of order $k+2$ at p.