ANALYSIS ON FRACTALS AS INFINITESIMAL LIMITS OF QUANTUM GRAPHS

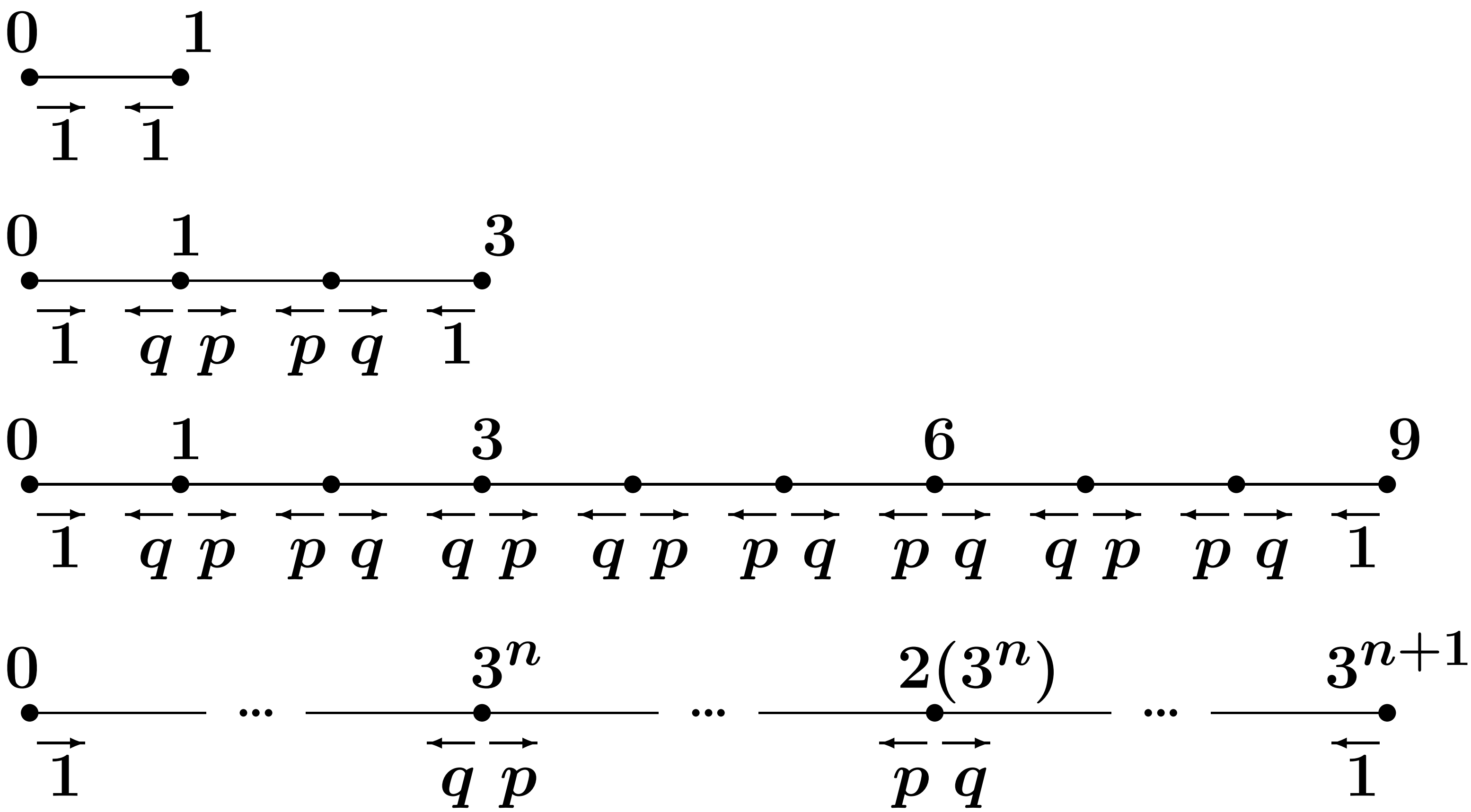
Alexander Teplyaev
University of Connecticut

Operator Theory, Analysis and Mathematical Physics OTAMP 2006

Lund, SWEDEN, June 15-22, 2006

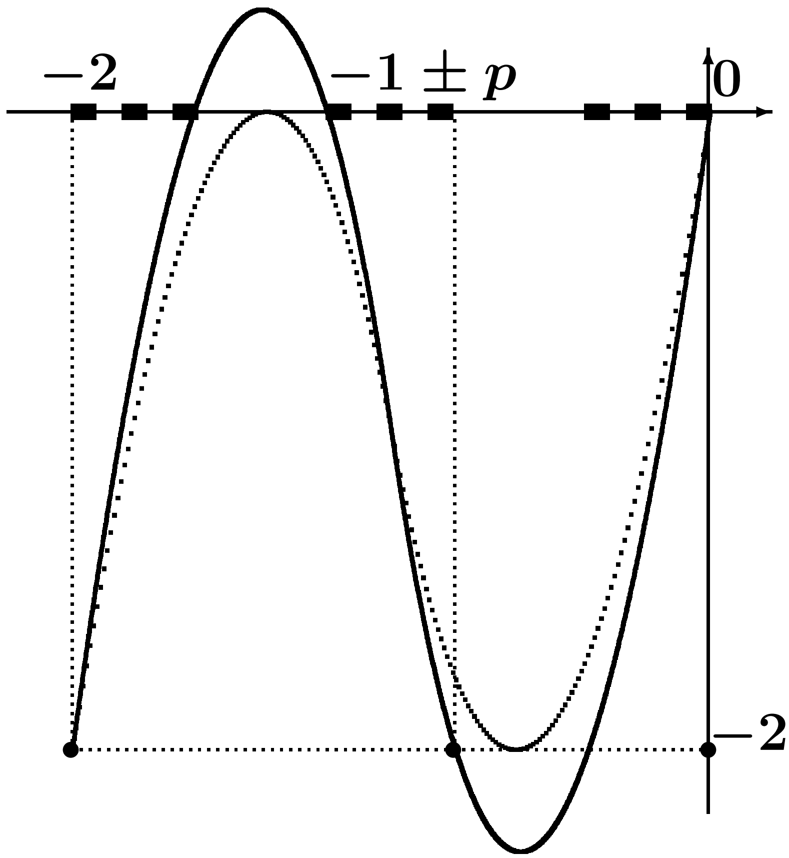
FRACTAL LATTICES

Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians Δ_n on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:



and let $\Delta = \lim_{n\to\infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$.

If $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$, then $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$



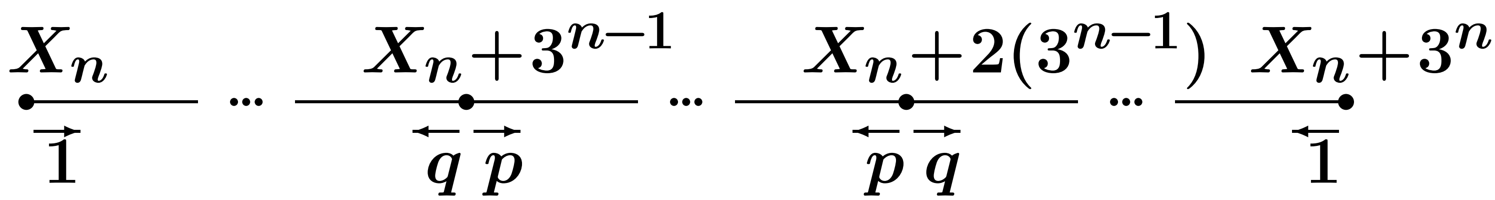
Theorem. $\sigma(\Delta) = \mathcal{J}_R$, the Julia set of $R(z)$.
If $p=q$, then $\sigma(\Delta)=[-2,0]$, spectrum is a.c.
If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.

There are uncountably many "random" self-similar Laplacians $\boldsymbol{\Delta}$ on $\mathbb{Z}$:

For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0, 1, 2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

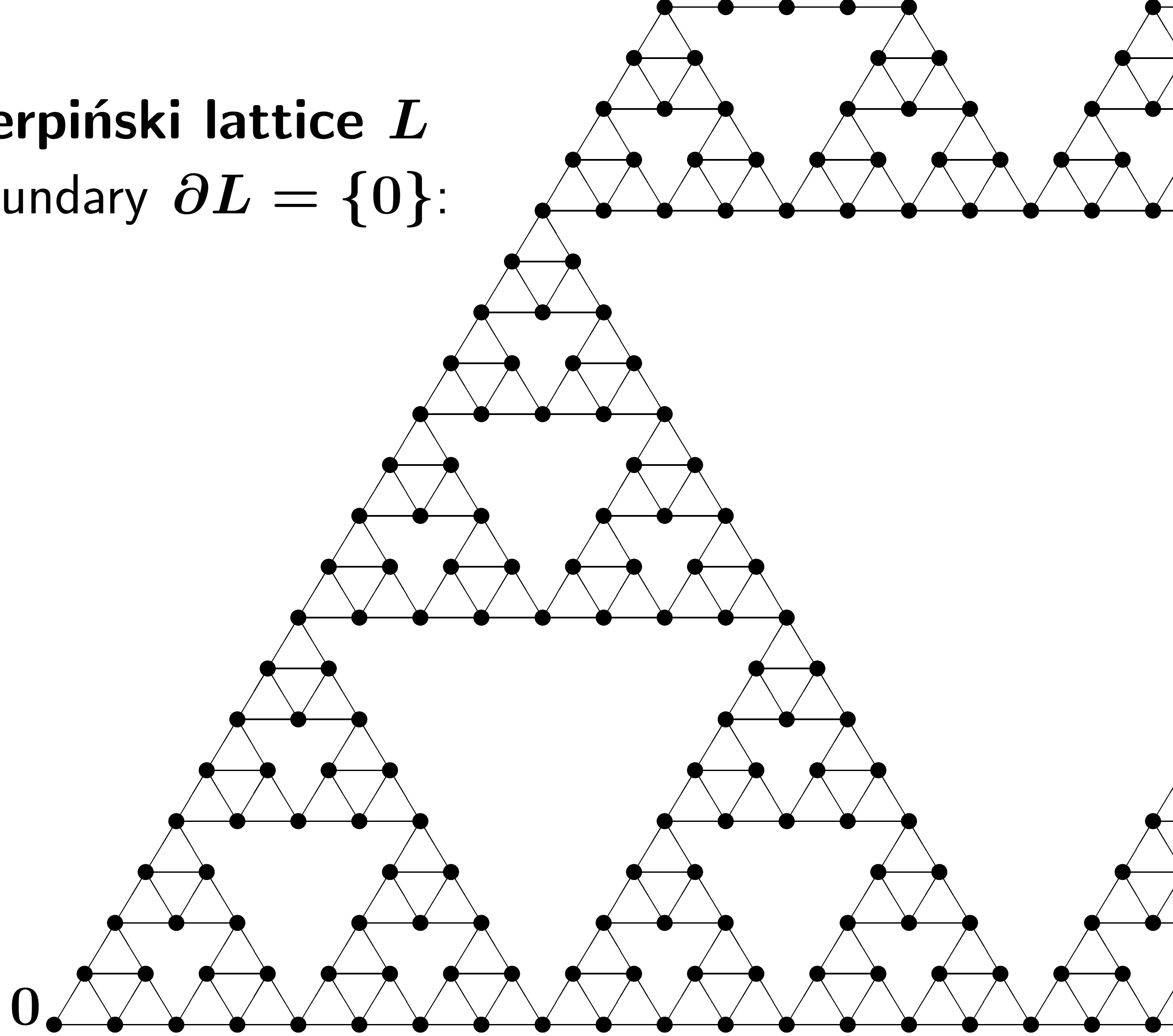
and $\boldsymbol{\Delta}_n$ is a probabilistic Laplacian on $[X_n, X_n + 3^n]$:



In the previous example $k_j = 0$ for all j.

Theorem.
For any sequence $\mathcal{K}$ we have $\sigma(\boldsymbol{\Delta}) = \mathcal{J}_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).

The **Sierpiński lattice** L with boundary $\partial L = \{0\}$:

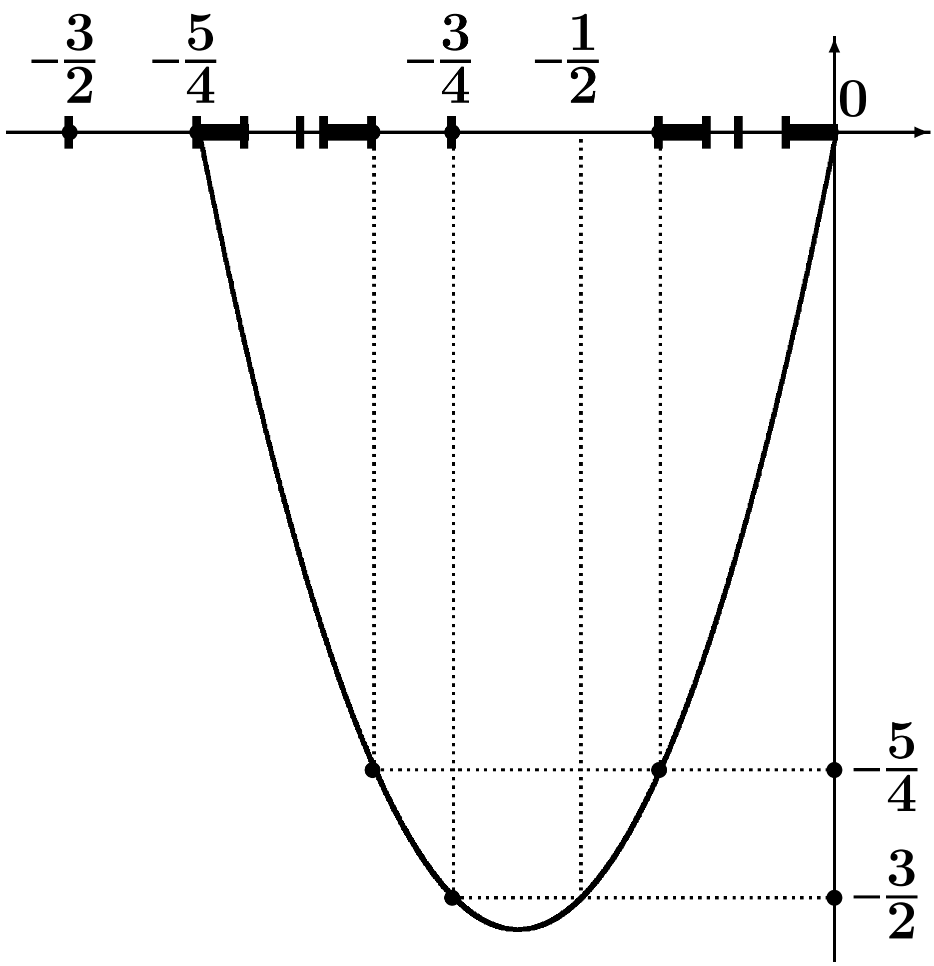


Let $\boldsymbol{\Delta}$ be the probabilistic Laplacian (generator of a simple random walk) on L.

Theorem (T). The spectrum of $\boldsymbol{\Delta}$ is pure point. Eigenfunctions with finite support are complete.

If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z+5)$, then

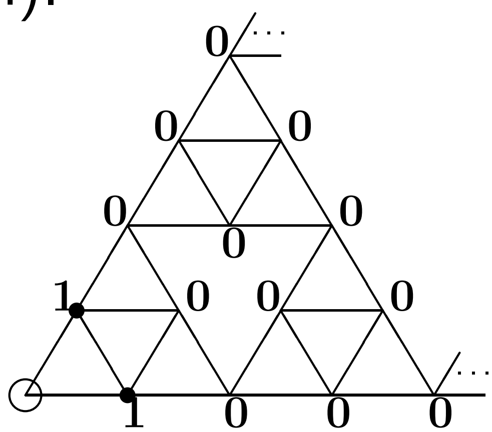
$$R(z) \in \sigma(\Delta) \iff z \in \sigma(\Delta)$$



$$\sigma(\Delta) = \mathcal{J}_R \cup \mathcal{D}$$

where $\mathcal{D} \stackrel{\text{def}}{=} \{-\frac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$ and $\mathcal{J}_R$ is the Julia set of $R(z)$.

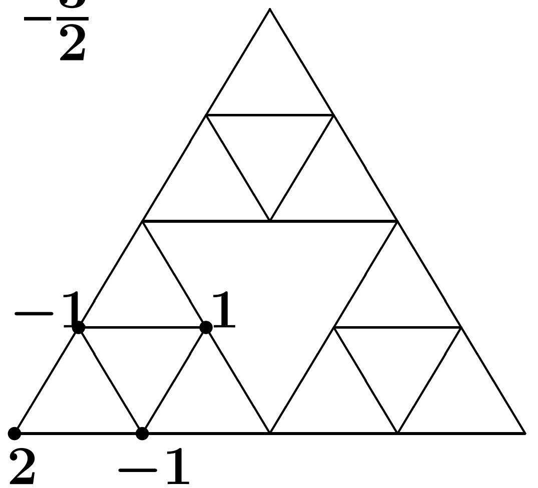
Let $\Delta^{(0)}$ be the Laplacian with zero (Dirichlet) boundary condition at ∂L. Then the compactly supported eigenfunctions of $\Delta^{(0)}$ are **not** complete (eigenvalues in $\mathcal{E}$ is not the whole spectrum).



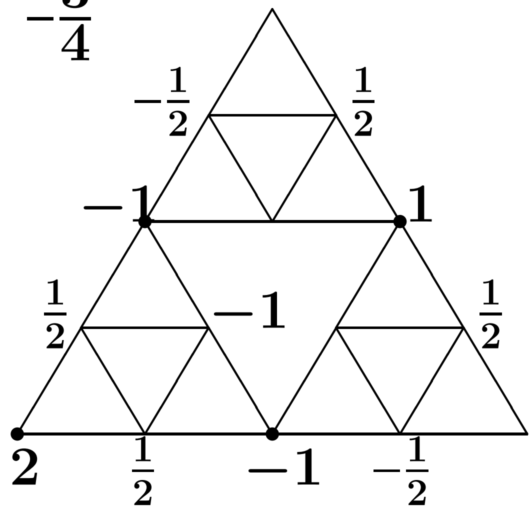
Let $\partial L^{(0)}$ be the set of two points adjacent to ∂L and $\omega_{\Delta}^{(0)}$ be the spectral measure of $\Delta^{(0)}$ associated with $1\!\!1_{\partial L^{(0)}}$. Then $\mathbf{supp}(\omega_{\Delta}^{(0)}) = \mathcal{J}_R$ has Lebesgue measure zero and

$$\frac{d(\omega_{\Delta}^{(0)} \circ R_{1,2})}{d\omega_{\Delta}^{(0)}}(z) = \frac{(8z+5)(2z+3)}{(2z+1)(4z+5)}$$

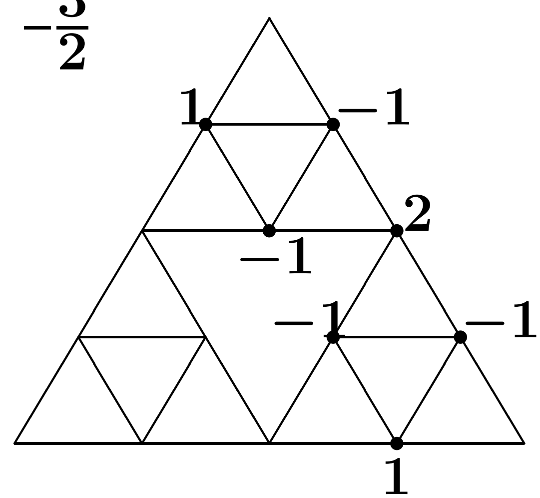
$z = -\frac{3}{2}$



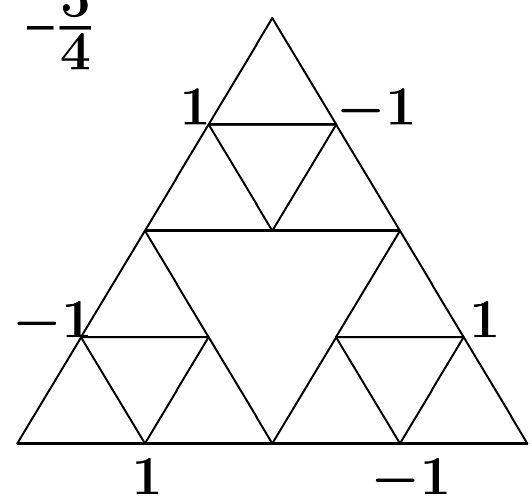
$z = -\frac{3}{4}$

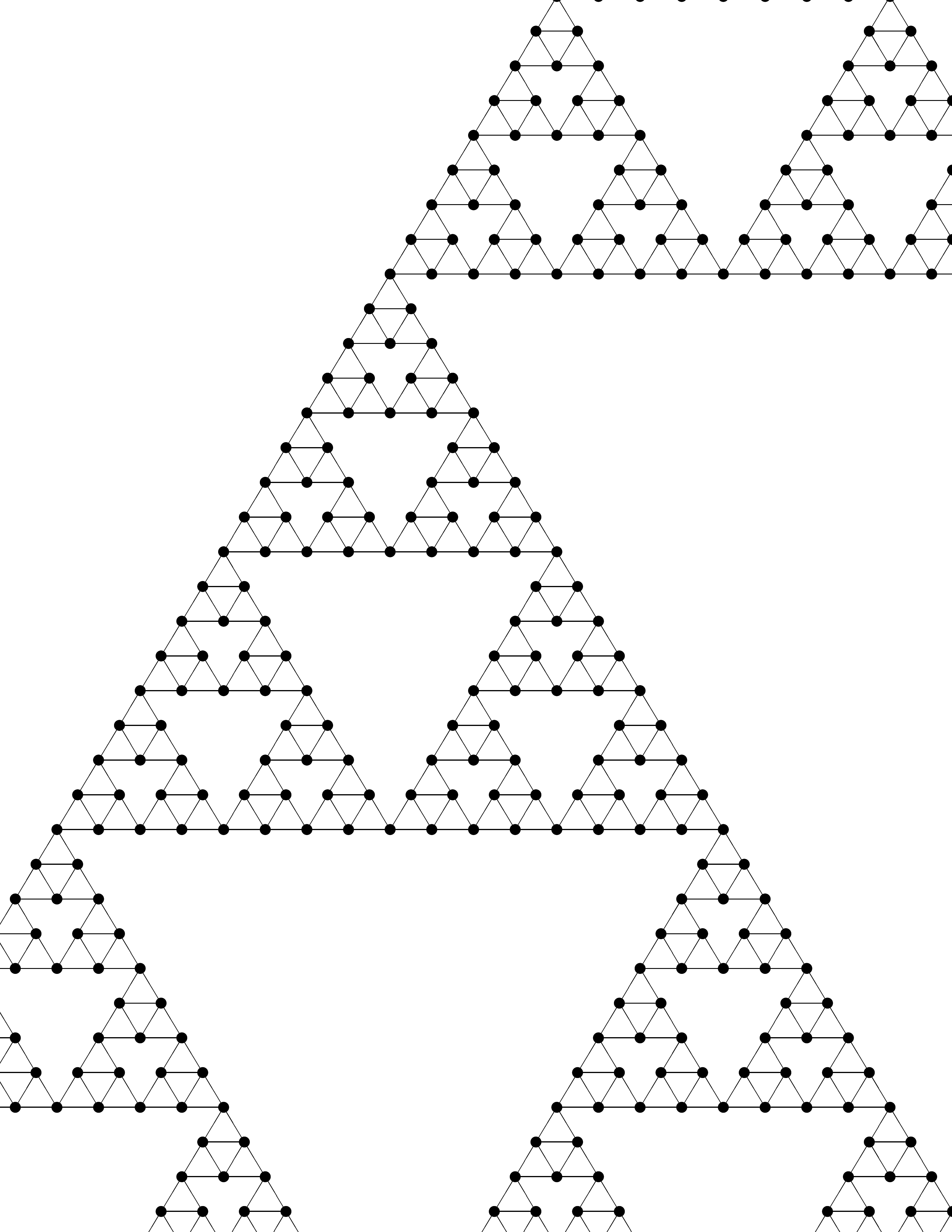


$z = -\frac{3}{2}$



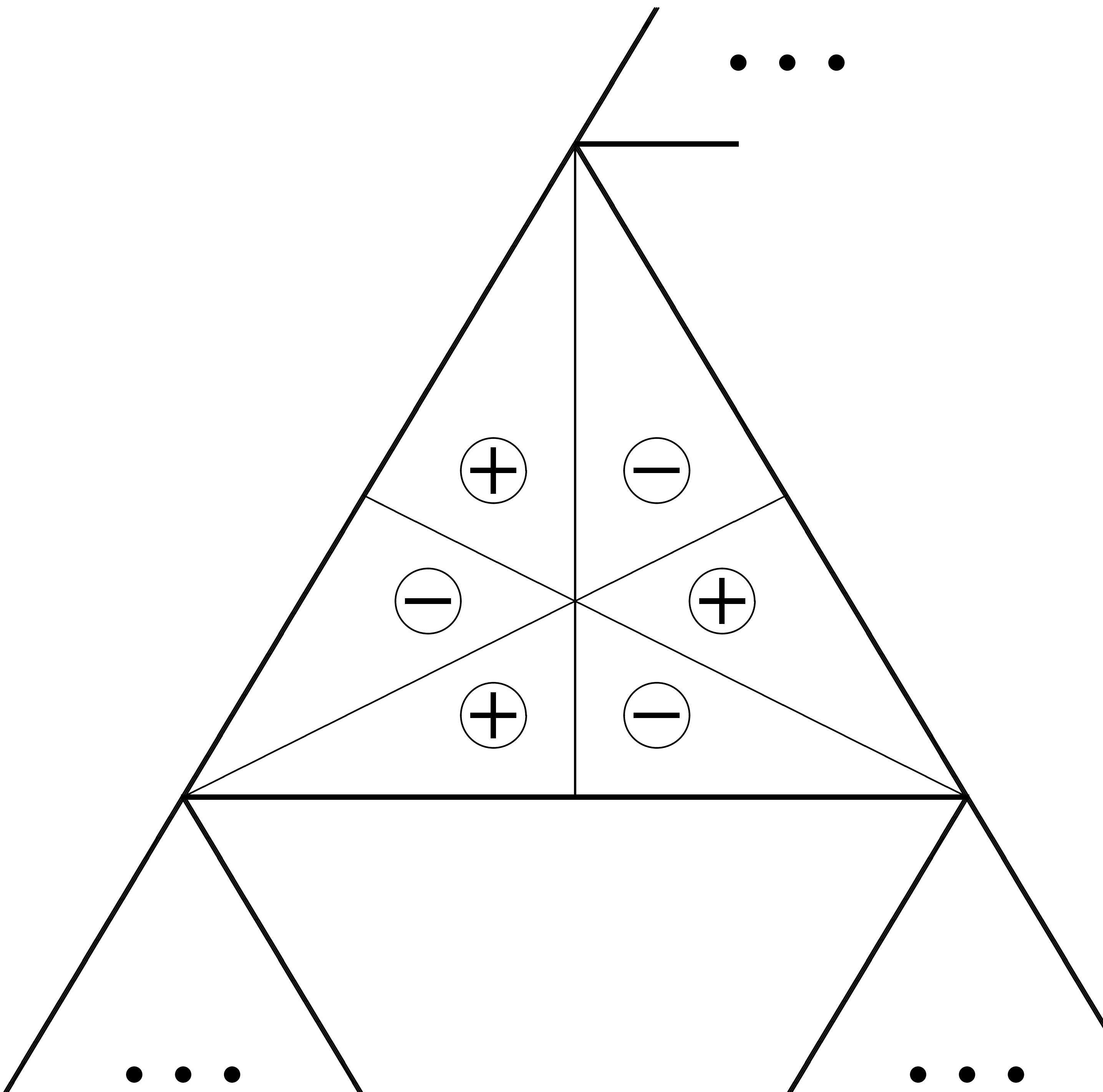
$z = -\frac{5}{4}$





There are uncountably many nonisomorphic Sierpiński lattices with empty boundary.

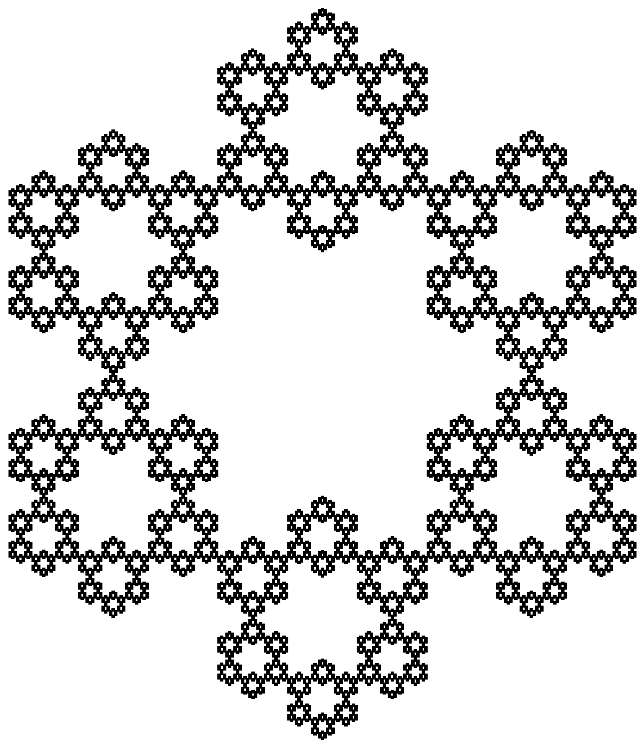
Theorem. If a Sierpiński lattice has no boundary, then the spectrum of $\boldsymbol{\Delta}$ is pure point, eigenfunctions with finite support are complete, the set of eigenvalues is $\mathcal{E}$.

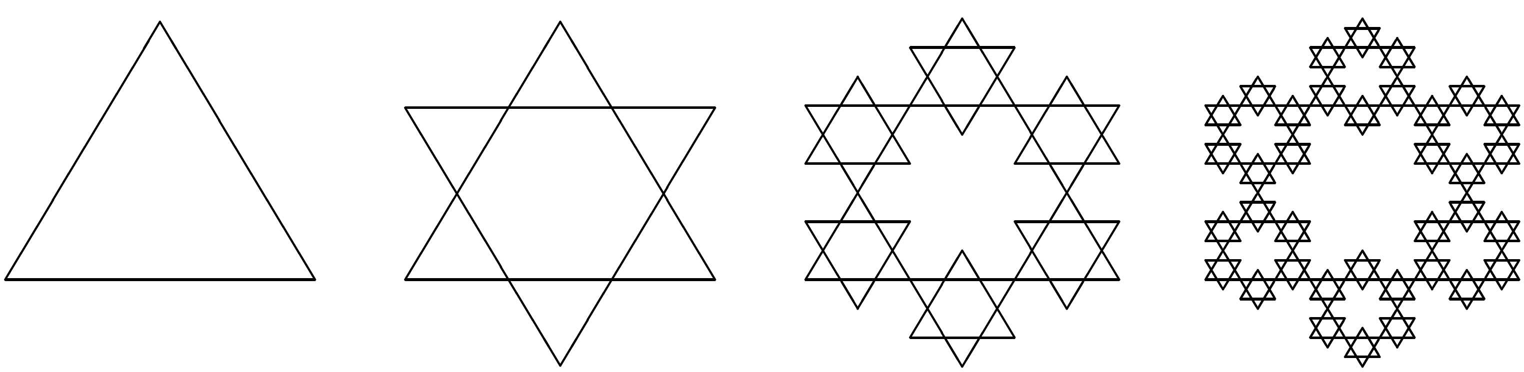


HEXAGASKET

or

FRACTAL STAR OF DAVID





$$R(z) = \frac{2z(z+1)(16z^2+24z+7)}{(2z+1)}$$

HARMONIC STRUCTURES ON FINITELY RAMIFIED SELF-SIMILAR FRACTALS

Definition. A compact connected metric space F is called a *finitely ramified self-similar set* if there are injective contraction maps $\psi_1, ..., \psi_m : F \longrightarrow F$ and a finite set $V_0 \subset F$ such that

$$F = \bigcup_{i=1}^{m} \psi_i(F)$$

and

$$F_w \cap F_{w'} = V_w \cap V_{w'}$$

for any two distinct words $w, w' \in W_n = \{1, ..., m\}^n$, where

$$F_w = \psi_w(F)$$
$$V_w = \psi_w(V_0)$$
$$\psi_w = \psi_{w_1} \circ ... \circ \psi_{w_n}$$

Definition. The group G of acts on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geqslant 0$.

Definition. A resistance form $\mathcal{E}$ is self-similar if

$$\mathcal{E}(f, f) = \sum_{i=1}^{m} \rho_i \mathcal{E}(f \circ \psi_i, f \circ \psi_i).$$

Proposition. Suppose a group G of acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permutation group of V_0. Then there exists a unique, up to a constant, G-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights ρ_i and

$$\mathcal{E}_0(f, f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2.$$

Theorem (Hambly, Metz, T.). Suppose that F has connected interior, and a group G acts on a self-similar finitely ramified fractal F such that its action on V_0 is transitive. Then there exists a G-invariant self-similar resistance form $\mathcal{E}$.

HARMONIC COORDINATES ON (POSSIBLY NON-SELF-SIMILAR) FRACTALS WITH FINITELY RAMIFIED CELL STRUCTURE

Definition. A pair $(\mathcal{E}, \operatorname{Dom}\mathcal{E})$ is a resistance form on a countable set V_* if

- $\operatorname{Dom}\mathcal{E}$ is a linear subspace of $\ell(V_*)$ containing constants, $\mathcal{E}$ is a nonnegative symmetric quadratic form on $\operatorname{Dom}\mathcal{E}$, and $\mathcal{E}(u,u)=0$ if and only if u is constant.
- Let $\sim$ be an equivalence relation on $\operatorname{Dom}\mathcal{E}$ defined by $u \sim v$ if and only if $u - v$ is constant on V_*.
  Then $(\mathcal{E}/\sim, \operatorname{Dom}\mathcal{E})$ is a Hilbert space.
- For any finite subset $V \subset V_*$ and for any $v \in \ell(V)$ there exists $u \in \operatorname{Dom}\mathcal{E}$ such that $u\big|_V = v$.
- (the Markov property)
- For any $p, q \in V_*$

$$R(p,q) = \sup\left\{\frac{\big(u(p)-u(q)\big)^2}{\mathcal{E}(u,u)} : u \in \operatorname{Dom}\mathcal{E}\right\} < \infty$$

$R(p,q)$ is the effective resistance between metric.

Any $u \in \operatorname{Dom}\mathcal{E}$ has a unique R-Hölder continuous extension to Ω, the R-completion of V_*.

Markov property: for any $\boldsymbol{u} \in \operatorname{Dom} \mathcal{E}$ we have that

$$\mathcal{E}(\bar{u}, \bar{u}) \leqslant \mathcal{E}(u, u),$$

where

$$\bar{u}(p) = \begin{cases} 1 & \text{if } u(p) \geqslant 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leqslant 1. \end{cases}$$

For any finite subset $U \subset V_*$ the finite dimensional Dirichlet form $\mathcal{E}_U$ on U is defined by

$$\mathcal{E}_U(f,f) = \inf\{\mathcal{E}(g,g) : g \in \operatorname{Dom}\mathcal{E}, g\big|_U = f\}$$

and is called the **trace** of $\mathcal{E}$ on U.

If $U_1 \subset U_2$ then $\mathcal{E}_{U_1}$ is the trace of $\mathcal{E}_{U_2}$ on U_1.

Theorem (Kigami). Suppose that V_n are finite subsets of V_* and that $\bigcup_{n=0}^{\infty} V_n$ is R-dense in V_*. Then

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_{V_n}(f,f)$$

for any $f \in \operatorname{Dom}\mathcal{E}$, where the limit is non-decreasing.

Theorem (Kigami). Suppose that V_n are finite sets, and the finite dimensional resistance forms $\mathcal{E}_{V_n}$ on V_n are compatible: each $\mathcal{E}_{V_n}$ is the trace of $\mathcal{E}_{V_{n+1}}$ on V_n.

Then there exists a resistance form $\mathcal{E}$ on $V_* = \bigcup_{n=0}^{\infty} V_n$ such that

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \mathcal{E}_{V_n}(f,f)$$

for any $f \in \operatorname{Dom}\mathcal{E}$, and the limit is non-decreasing.

Definition. A *finitely ramified fractal* F is a compact metric space with a *cell structure* $\mathcal{F} = \{F_\alpha\}_{\alpha\in\mathcal{A}}$ and a *boundary (vertex) structure* $\mathcal{V} = \{V_\alpha\}_{\alpha\in\mathcal{A}}$ such that the following conditions hold.

- $\mathcal{A}$ is a countable index set;
- each F_α is a distinct compact connected subsets of F;
- each V_α is a finite subset of F_α with at least two elements;
- if $F_\alpha = \bigcup_{j=1}^{k} F_{\alpha_j}$ then $V_\alpha \subset \bigcup_{j=1}^{k} V_{\alpha_j}$;
- there exists a filtration $\{\mathcal{A}_n\}_{n=0}^{\infty}$ such that
  (1) $\mathcal{A}_n$ are finite subsets of $\mathcal{A}$, $\mathcal{A}_0 = \{0\}$, $F_0 = F$;
  (2) $\mathcal{A}_n \cap \mathcal{A}_m = \varnothing$ if $n \neq m$;
  (3) for any $\alpha \in \mathcal{A}_n$ there are $\alpha_1, ..., \alpha_k \in \mathcal{A}_{n+1}$ such that $F_\alpha = \bigcup_{j=1}^{k} F_{\alpha_j}$;
- for any two distinct $\alpha, \alpha' \in \mathcal{A}_n$

$$F_{\alpha'} \bigcap F_\alpha = V_{\alpha'} \bigcap V_\alpha;$$

- for any strictly decreasing infinite sequence of cells there exists $x \in F$ such that $\bigcap_{n\geqslant 1} F_{\alpha_n} = \{x\}$.

If these conditions are satisfied, then

$$(F, \mathcal{F}, \mathcal{V}) = (F, \{F_\alpha\}_{\alpha\in\mathcal{A}}, \{V_\alpha\}_{\alpha\in\mathcal{A}})$$

is called a *finitely ramified cell structure.*

Definition. A function is harmonic if it minimizes the energy for the given set of boundary values.

A function is n-harmonic if it minimizes the energy for the given set of values on V_n.

Theorem. Suppose that all n-harmonic functions are continuous. Then any continuous function is R-continuous, and any R-Cauchy sequence converges in the topology of F. Also, there is a continuous injective map $\theta : \Omega \longrightarrow F$ which is the identity on V_*.

Then we can (and will) consider Ω as a subset of F. Then Ω is the R-closure of V_*. In a sense, Ω is the set where the Dirichlet form $\mathcal{E}$ "lives".

Theorem. Suppose that all n-harmonic functions are continuous. Then $\mathcal{E}$ is a local regular Dirichlet form on Ω (with respect to any measure that charges every nonempty open set).

Definition. We fix a complete, up to constant functions, energy orthonormal set of harmonic functions $h_1, ..., h_k$, where $k = |V_0| - 1$, and define the Kusuoka energy measure by

$$\nu = \nu_{h_1} + ... + \nu_{h_k}.$$

If $F_{\alpha'} \subset F_\alpha$, then

$$M_{\alpha,\alpha'} : \ell(V_\alpha) \to \ell(V_{\alpha'})$$

is the linear map which is defined as follows. If f_α is a function on V_α then let h_{f_α} be the unique harmonic function on F_α that coincides with f_α on V_α. Then we define

$$M_{\alpha,\alpha'} f_\alpha = h_{f_\alpha}\big|_{V_{\alpha'}}.$$

We denote $M_\alpha = M_{0,\alpha}$. We denote D_α the matrix of the Dirichlet form $\mathcal{E}_\alpha$ on V_α.

Proposition. If $F_\alpha = \bigcup F_{\alpha_j}$ then

$$D_\alpha = \sum M^*_{\alpha,\alpha_j} D_{\alpha_j} M_{\alpha,\alpha_j}$$

and

$$\nu(F_\alpha) = \operatorname{Tr} M^*_\alpha D_\alpha M_\alpha.$$

We denote

$$Z_\alpha = \frac{M_\alpha^* D_\alpha M_\alpha}{\nu(F_\alpha)}$$

if $\nu(F_\alpha) \neq 0$. Then we define matrix valued functions

$$Z_n(x) = Z_\alpha$$

if $\nu(F_\alpha) \neq 0$, $\alpha \in \mathcal{A}_n$ and $x \in F_\alpha \backslash V_\alpha$. Note that $\operatorname{Tr} Z_n(x) = 1$ by definition.

Theorem. For ν-almost all x there is a limit

$$Z(x) = \lim_{n \to \infty} Z_n(x).$$

Proof. One can see, following Kusuoka's idea, that Z_n is a bounded ν-martingale. □

The energy measures ν_h are the same as the energy measures in the general theory of Dirichlet forms. The matrix Z is the matrix whose entries are the densities

$$Z_{ij} = \frac{d\nu_{h_i,h_j}}{d\nu}$$

Theorem. If the space of piecewise harmonic functions is dense in $\mathrm{Dom}\,\mathcal{E}$ then any $f \in \mathrm{Dom}\,\mathcal{E}$ has a weak gradient ∇f such that

$$\mathcal{E}(f,f) = \int_F \langle \nabla f, Z \nabla f \rangle d\nu$$

Conjecture: for any finitely ramified fractal

$$\mathrm{rank} Z(x) = 1$$

for ν-almost all x.

GRADIENT IN HARMONIC COORDINATED

Let $V_0 = \{v_1, ..., v_m\}$ and let h_j be the unique harmonic function with boundary values $h_j(v_i) = \delta_{i,j}$.

Kigami's harmonic coordinate map $\psi : F \longrightarrow \mathbb{R}^m$ is

$$\psi(x)=(h_1(x), ..., h_m(x)).$$

In what follows we assume that $\psi : F \longrightarrow F_H = \psi(F)$ *is a homeomorphism,* $F = F_H$*,* $\psi(x) = x$ *and identify* $\ell(V_0)$ *with* $\mathbb{R}^m$ *in the natural way.*

Theorem. If f is the restriction to F of a $C^1(\mathbb{R}^m)$ function then $f \in \operatorname{Dom}\mathcal{E}$, and such functions are dense in $\operatorname{Dom}\mathcal{E}$. Moreover,

$$\mathcal{E}(f, f) = \int_F \langle \nabla f, Z\nabla f \rangle d\nu$$

for any $f \in C^1(\mathbb{R}^m)$.

ENERGY MEASURE Laplacian IN HARMONIC COORDINATES

We have the analog of the Gauss-Green formula:

$$\mathcal{E}(f,g) = -\int_F g\Delta_\nu f d\nu,$$

for any function $g \in \operatorname{Dom}\mathcal{E}$, vanishing on the boundary V_0, and any function $f \in \operatorname{Dom}\Delta_\nu$, where Δ_ν is the **energy Laplacian**.

Theorem. If f is the restriction to F of a $C^2(\mathbb{R}^m)$ function then $f \in \operatorname{Dom}\Delta_\nu$, and such functions are dense in $\operatorname{Dom}\Delta_\nu$. Moreover, ν-almost everywhere

$$\Delta_\nu f = \operatorname{Tr}(ZD^2 f)$$

where $D^2 f$ is the matrix of the second derivatives of f.

Conjecture: if $f \in \operatorname{Dom}\Delta_\nu$ then f is the restriction to F of a $C^1(\mathbb{R}^m)$.