Orthogonality of multi-variable polynomials

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talk based on the paper:

D. Cichoń, J. Stochel, F. H. Szafraniec, Three term recurrence relation modulo ideal and orthogonality of polynomials of several variables, *Journal of Approximation Theory* **134** (2005), 11-64.

Orthogonal polynomials - single variable.

Theorem 1 (Favard's theorem) *If* $\{p_k\}_{k=0}^{\infty}$ *is a sequence of real polynomials in a single variable such that* $p_0 = 1$*, then* **TFAE:**

(i) $\deg p_k = k$ *for all* $k \geqslant 0$ *and there exists a positive definite linear functional* $L : \mathbb{C}[X] \to \mathbb{C}$ *(equivalently: a Borel measure* $\mu \geqslant 0$ *on* $\mathbb{R}$*) which orthonormalizes* $\{p_k\}_{k=0}^{\infty}$*, i.e.*

$$L(p_k p_l) = \delta_{k,l}, \quad k, l \geqslant 0,$$
$$(equivalently: \int_{\mathbb{R}} p_k p_l \, \mathrm{d}\mu = \delta_{k,l}, \quad k, l \geqslant 0,)$$

(ii) *for every* $k \geqslant 0$*, there exist* $a_k \in \mathbb{R} \setminus \{0\}$ *and* $b_k \in \mathbb{R}$ *such that*

$$X p_k = a_k p_{k+1} + b_k p_k + a_{k-1} p_{k-1},$$

where $a_{-1} \stackrel{\mathrm{df}}{=} 1$ *and* $p_{-1} \stackrel{\mathrm{df}}{=} 0$.

If **(i)** *holds, then the closed support* $\operatorname{supp} \mu$ *of* μ *is infinite.*

Several variables: much more complex.

The very first difficulty:

convenient notation

- related to the order in which the orthonormalization procedure has to be performed
- allowing us to see the recurrence relation as a three term one.

The pioneering attempt: M. A. Kowalski (1982).

Continuation:

Y. Xu (1993, 1994)

M. I. Gekhtman and A. A. Kalyuzhny (1991, 1994).

Further difference:

- the three term recurrence relation (regardless the way the relation is built up) may not determine any orthogonality measure though the functional orthogonality in Favard's Theorem is still preserved.

Lack:

- the three term recurrence relation considered so far forces the Zariski closure of the support of an orthogonalizing measure, provided it exists, to be the whole space $\mathbb{R}^N$.
- important instances (e.g. sphere) are left out.

Aim: study the case of orthogonalizing measures **not having too massive supports**.

Our point of view : recurrence relations of matrix type satisfied **modulo an ideal**.

Notation and definitions:
$\mathbb{C}$ = complex numbers; $\mathbb{R}$ = real numbers; $\mathbb{N}$={0,1,2, ...};

$$\mathcal{P}_N \overset{\text{df}}{=} \mathbb{C}[X_1, \dots, X_N], \quad \mathcal{R}_N \overset{\text{df}}{=} \mathbb{R}[X_1, \dots, X_N],$$
$$\mathcal{P}_N^{\langle k]} \overset{\text{df}}{=} \{p \in \mathcal{P}_N : \deg p \leqslant k\}, \quad k \in \mathbb{N},$$

$\deg p$ = the degree of a polynomial p.

A sequence $\{P_k\}_{k=0}^{\infty}$ of column polynomials is said to be a **rigid basis** of $\mathcal{P}_N$ if $\{P_k\}_{k=0}^{\infty}$ is a **column representation** of a (linear) basis of $\mathcal{P}_N$ such that for every $k \geqslant 0$, $P_k \subseteq \mathcal{P}_N^{\langle k]}$ and

$$\ell(P_k) = \text{card}\{\alpha \in \mathbb{N}^N : |\alpha| = k\} = \binom{k+N-1}{k},$$

where $\ell(P_k)$ is the **length** of P_k (i.e. the number of entries of P_k).

By a **rigid basis** of $\mathcal{R}_N$ we mean a (linear) basis $\{P_k\}_{k=0}^{\infty}$ of $\mathcal{R}_N$ which is simultaneously a rigid basis of $\mathcal{P}_N$.

Monomials

$$X^\alpha = X_1^{\alpha_1} \cdots X_N^{\alpha_N}, \quad \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N,$$

form a rigid basis of $\mathcal{R}_N$ as well as $\mathcal{P}_N$.

Theorem 2 Y. Xu *(1993) Let* $\{P_k\}_{k=0}^{\infty}$ *be a sequence of* real column polynomials *such that* $P_0 \neq 0$*. Then* **TFAE:**

(i) $\{P_k\}_{k=0}^{\infty}$ *is a* rigid basis *of* $\mathcal{R}_N$ *for which there exists a linear functional* $L : \mathcal{R}_N \to \mathbb{R}$ *such that* $L(P_i P_j^{\mathsf{T}}) = 0$ *for all* $i \neq j$ *and* $L(P_k P_k^{\mathsf{T}})$ *is a non-singular matrix for every* $k \geqslant 0$*;*

(ii) *for every* $k \geqslant 0$*, there exists a system* $A_{k,1}, \ldots, A_{k,N}$*,* $B_{k,1}, \ldots, B_{k,N}$*,* $C_{k,1}, \ldots, C_{k,N}$ *of scalar real matrices such that*

(ii-a) $X_j P_k = A_{k,j} P_{k+1} + B_{k,j} P_k + C_{k,j} P_{k-1}$ *for all* $j = 1, \ldots, N$*, where* $C_{0,j} \stackrel{\text{df}}{=} 1$ *and* $P_{-1} \stackrel{\text{df}}{=} 0$*,*

(ii-b) *the length of* $P_k \leqslant \binom{k+N-1}{k}$*,*

(ii-c) $\deg P_k \leqslant k$, $\deg P_k \stackrel{\text{df}}{=} \max\{\deg p \colon p \in P_k\}$

(ii-d) *the matrix* $[C_{k,1}, \ldots, C_{k,N}]$ *is of maximal rank.*

If **(i)** *holds, then for every* $p \in \mathcal{R}_N$*:*

$$p = 0 \text{ if and only if } L(pq) = 0 \text{ for all } q \in \mathcal{R}_N.$$

If **(ii)** *holds, then the matrix* $[A_{k,1}^{\mathsf{T}}, \ldots, A_{k,N}^{\mathsf{T}}]^{\mathsf{T}}$ *is of maximal rank for all* $k \geqslant 0$*.*

Notation and definitions:
Let V be a proper ideal in $\mathcal{P}_N \overset{\text{df}}{=} \mathbb{C}[X_1, \ldots, X_N]$.

If P and Q are column polynomials of the same length and each entry of $P - Q$ belongs to V, then we write $P \overset{V}{=} Q$.

Fix an integer $m \geqslant 1$. A sequence of column polynomials $\{Y_k\}_{k=0}^{n}$ with entries in $\mathcal{P}_N$ $(0 \leqslant n \leqslant \infty)$ is said to be a V-**basis of** $\mathcal{P}_N$ if for every column polynomial P of length m, there exists a unique finite sequence $\{A_k\}_{k=0}^{n}$ of scalar matrices with m rows such that

$$P \overset{V}{=} \sum_{k=0}^{n} A_k Y_k.$$

The above definition is independent of m :

Notation and definitions:

V is a proper ideal in $\mathcal{P}_N$,

$\mathit{\Pi}_V : \mathcal{P}_N \longrightarrow \mathcal{P}_N/V$ is the quotient mapping:

$$\mathit{\Pi}_V(p) \overset{\text{df}}{=} p + V, \quad p \in \mathcal{P}_N.$$

We define the sequence $\{d_V(k)\}_{k=0}^{\infty}$:

$$d_V(k) = \begin{cases} \dim \mathit{\Pi}_V\Big(\mathcal{P}_N^{\langle 0]}\Big) = 1, & k = 0, \\ \dim \mathit{\Pi}_V\Big(\mathcal{P}_N^{\langle k]}\Big) - \dim \mathit{\Pi}_V\Big(\mathcal{P}_N^{\langle k-1]}\Big), & k \geqslant 1, \end{cases}$$

and $\varkappa_V$:

$$\varkappa_V = \sup\{j \geqslant 0 : d_V(j) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Properties of $\{d_V(k)\}_{k=0}^{\infty}$:

$$d_V(k+j) \leqslant N^j d_V(k), \quad k, j \geqslant 0,$$

$$\{j \in \mathbb{N} \colon d_V(j) \neq 0\} = \{j \in \mathbb{N} \colon j \leqslant \varkappa_V\} = \overline{0, \varkappa_V},$$

$$\varkappa_V = \dim \mathcal{P}_N/V.$$

Notation and definitions:

V is a proper ideal in $\mathcal{P}_N$,

A V-basis $\{P_k\}_{k=0}^n$ of $\mathcal{P}_N$ is called **rigid** if

$$n = \varkappa_V,$$
$$P_k \subseteq \mathcal{P}_N^{\langle k]}, \quad k \in \overline{0, \varkappa_V},$$
$$\ell(P_k) = d_V(k), \quad k \in \overline{0, \varkappa_V}.$$

If $\{P_k\}_{k=0}^{\varkappa_V}$ is a rigid basis of $\mathcal{P}_N$, then the degree of each member of P_k is equal to k .
If $V = \{0\}$, then we get the definition of a rigid basis of $\mathcal{P}_N$.

There exists a special rigid V-basis $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ composed of monomials, which plays the role of the standard basis of monomials in $\mathbb{C}[X_1, \ldots, X_N]$. In particular, we have:

$$\Sigma_{k+1}^V \subseteq \bigcup_{j=1}^N X_j \Sigma_k^V,$$

here Σ_k^V and Σ_{k+1}^V are interpreted as sets.

Construction of $\{\Sigma_k^V\}_{k=0}^{\infty}$.

Define a total order $\lessdot$ on $\mathbb{N}^N$ by:

$\alpha \underline{\lessdot} \beta$ if and only if either $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and α precedes β with respect to the lexicographic order.

Then write $\alpha \lessdot \beta$ if $\alpha \underline{\lessdot} \beta$ and $\alpha \neq \beta$.

Recurrence definition:

- $\Sigma_0^V = \{X^0\}$,
- the monomial X^α belongs to Σ_{k+1}^V if and only if $X^\alpha \in \bigcup_{j=1}^N X_j \Sigma_k^V$ and

$$\Pi_V(X^\alpha) \notin \text{lin}\, \Pi_V\Big(\{X^0\} \cup \Big\{X^\beta \in \bigcup_{i=0}^{k} \bigcup_{j=1}^{N} X_j \Sigma_i^V : \beta \lessdot \alpha\Big\}\Big).$$

Another important properties of $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$:

Lemma 3 *If V is a proper ideal in $\mathcal{P}_N$, then for every $k \in \overline{1, \varkappa_V}$, there exists a column polynomial R_k and an injective scalar matrix M_k such that*

$$\begin{bmatrix} X_1 \Sigma_{k-1}^V \\ \vdots \\ X_N \Sigma_{k-1}^V \end{bmatrix} \overset{V}{=} M_k \Sigma_k^V + R_k \text{ and } \deg R_k < k. \quad (1)$$

The matrix M_k appearing in (1) is unique.

Proposition 4 *Let V be a proper ideal in $\mathcal{P}_N$. A sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of column polynomials is a rigid V-basis of $\mathcal{P}_N$ if and only if the following two conditions hold for every $k \in \overline{0, \varkappa_V}$,*

1° $\deg q \leqslant k$ *for every entry q of Q_k,*

2° there exists a non-singular scalar matrix G_k and a column polynomial R_k such that

$$Q_k \overset{V}{=} G_k \Sigma_k^V + R_k \text{ and } \deg R_k < k.$$

Notation and definitions:

We equip the algebra $\mathcal{P}_N$ with a unique involution $p \mapsto p^*$ such that $X_i^* = X_i$ for all $i = 1, \dots, N$, i.e.

$$\Big(\sum_{\alpha \in \mathbb{N}^N} a_\alpha X^\alpha\Big)^* = \sum_{\alpha \in \mathbb{N}^N} \bar{a}_\alpha X^\alpha.$$

If $P = [p_{kl}]_{k=1}^{m}{}_{l=1}^{n}$ with $p_{kl} \in \mathcal{P}_N$, then

$$P^* = [q_{kl}]_{k=1}^{n}{}_{l=1}^{m} \text{ with } q_{kl} = p_{lk}^*.$$

A linear functional $L : \mathcal{P}_N \to \mathbb{C}$ can also be considered as a mapping operating on matrix polynomials via

$$L([p_{kl}]_{k=1}^{m}{}_{l=1}^{n}) \stackrel{\text{df}}{=} [L(p_{kl})]_{k=1}^{m}{}_{l=1}^{n}, \quad p_{kl} \in \mathcal{P}_N.$$

For simplicity of notation we do not indicate the dependence of so-defined mappings on the size of matrices.

A linear functional $L : \mathcal{P}_N \to \mathbb{C}$ is said to be **Hermitian** if $L(p^*) = \overline{L(p)}$ for all $p \in \mathcal{P}_N$.

Given a linear functional $L : \mathcal{P}_N \to \mathbb{C}$, we set

$$\mathcal{V}_L = \bigcap_{q \in \mathcal{P}_N} \{p \in \mathcal{P}_N : L(pq) = 0\}.$$

$\mathcal{V}_L$ is the greatest ideal contained in $\ker L$;

$\mathcal{V}_L$ is a proper ideal if and only if L is nonzero.

If L is a Hermitian linear functional, then $\mathcal{V}_L$ is a $*$-ideal (not conversely).

Given a $*$-ideal V in $\mathbb{C}[X_1, \ldots, X_N]$, we set

$$\Xi_k^V = \begin{bmatrix} \Sigma_0^V \\ \vdots \\ \Sigma_k^V \end{bmatrix}, \quad k \in \overline{0, \varkappa_V}.$$

Proposition 5 *Let $L : \mathcal{P}_N \to \mathbb{C}$ be a nonzero Hermitian linear functional and let $V = \mathcal{V}_L$. Then the following conditions are equivalent:*

(i) *there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ composed of real column polynomials such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$,*

(ii) *there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$,*

(iii) *the matrix $L(\Xi_k^V (\Xi_k^V)^*)$ is non-singular for every $k \in \overline{0, \varkappa_V}$,*

(iv) *rank $L(\Xi_k^{\{0\}} (\Xi_k^{\{0\}})^*) = d_V(0) + \ldots + d_V(k)$ for every integer $k \geqslant 0$.*

Theorem 6 *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L\colon \mathcal{P}_N \to \mathbb{C}$ be a linear functional and $\{Q_k\}_{k=0}^n$ be a sequence of real column polynomials such that $Q_0 \neq 0$ $(0 \leqslant n \leqslant \infty)$.*
Consider the following two conditions:

(A) $\{Q_k\}_{k=0}^n$ *is a rigid V-basis of $\mathcal{P}_N$,*
$L(Q_iQ_j^*) = 0$ *for all $i \neq j$* , *and* $V = \mathcal{V}_L$;

(B) $n = \varkappa_V$ *and there exist scalar matrices* $\{A_{k,j}\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$, $\{B_{k,j}\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$ *and* $\{C_{k,j}\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$ *such that*

(B-i) $X_jQ_k \overset{V}{=} A_{k,j}Q_{k+1} + B_{k,j}Q_k + C_{k,j}Q_{k-1}$
for all $j \in \overline{1,N}$ *and* $k \in \overline{0,\varkappa_V}$,

(B-ii) $\ell(Q_k) \leqslant d_V(k)$ *for every* $k \in \overline{0,\varkappa_V}$,

(B-iii) $\deg Q_k \leqslant k$ *for every* $k \in \overline{0,\varkappa_V}$,

(B-iv) *the matrix* $[C_{k,1},\dots,C_{k,N}]$ *is of maximal rank for every* $k \in \overline{0,\varkappa_V}$.

Under the above assumptions
the conditions (A) and (B) are equivalent.

$C_{0,j} \overset{\text{df}}{=} 1$ *and* $Q_{-1} \overset{\text{df}}{=} 0$;

if $\varkappa_V < \infty$, *then* $A_{\varkappa_V,j} \overset{\text{df}}{=} [1,\dots,1]^*$ *with* $\ell(A_{\varkappa_V,j}) = \ell(Q_{\varkappa_V})$;

$Q_{\varkappa_V+1} \overset{\text{df}}{=} 0$;

Moreover the following holds:

1° the conditions (B-i), (B-ii) and (B-iii) imply that the matrix $[A_{k,1}^*, \dots, A_{k,N}^*]^*$ is injective for every $k \in \overline{0, \varkappa_V}$, $\{Q_j\}_{j=0}^n$ is a rigid V-basis of $\mathcal{P}_N$, the linear functional L defined by

$$L|_V = 0,\ L(Q_0) = Q_0,\ L(Q_k) = 0\ \forall\ k \in \overline{1, \varkappa_V}, \tag{2}$$

is Hermitian and $L(Q_k Q_l^*) = 0$ for all $k \neq l$,

2° the condition (B) implies the non-singularity of $L(Q_k Q_k^*)$ for every $k \in \overline{0, \varkappa_V}$, where L is defined by (2),

3° if a linear functional $L' : \mathcal{P}_N \to \mathbb{C}$ satisfies (A), then $L'(X^0) \neq 0$ and the functional $\frac{1}{L'(X^0)} L'$ fulfils (2).

There is and interplay between rank and degree.

Orthogonality:

If $L : \mathcal{P}_N \to \mathbb{C}$ is a **positive definite** linear functional, i.e. $L(pp^*) \geqslant 0$ for all $p \in \mathcal{P}_N$, then

$$\mathcal{V}_L = \{p \in \mathcal{P}_N : L(pp^*) = 0\},$$

where $\mathcal{V}_L = \bigcap_{q \in \mathcal{P}_N} \{p \in \mathcal{P}_N : L(pq) = 0\}$.

Given a linear functional $L : \mathcal{P}_N \to \mathbb{C}$, we say that a sequence $\{Q_k\}_{k=0}^n$ of column polynomials ($0 \leqslant n \leqslant \infty$) is L-**orthonormal** if

$$L(Q_i Q_j^*) = 0, \quad i \neq j,$$
$$L(Q_k Q_k^*) = \text{ the identity matrix, } k \in \overline{0, n}.$$

Fact: Each L-orthonormal sequence $\{Q_k\}_{k=0}^n$ is linearly V-independent for any ideal $V \subseteq \mathcal{V}_L$.

Proposition 7 *If $L : \mathcal{P}_N \to \mathbb{C}$ is a nonzero linear functional, then* **TFAE:**

(i) *L is positive definite,*

(ii) *$\mathcal{V}_L$ is a $*$-ideal and there is a rigid $\mathcal{V}_L$-basis of $\mathcal{P}_N$ which is L-orthonormal,*

(iii) *$\mathcal{V}_L$ is a $*$-ideal and there is a $\mathcal{V}_L$-basis of $\mathcal{P}_N$ which is L-orthonormal,*

(iv) *there is a basis B of $\mathcal{P}_N$ such that $L(pp^*) \in \{0, 1\}$ and $L(qr^*) = 0$ for all $p, q, r \in B$ such that $q \neq r$.*

Remark: If we drop the assumption that $\mathcal{V}_L$ is a $*$-ideal in any of the conditions (ii) and (iii) of Proposition 7, then the functional L may not be positive definite.

Theorem 8 *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional and $\{Q_k\}_{k=0}^n$ be a sequence of real column polynomials such that $Q_0 = 1$ $(0 \leqslant n \leqslant \infty)$.*
Consider the following two conditions:

(A) *$\{Q_k\}_{k=0}^n$ is a rigid V-basis of $\mathcal{P}_N$, which is L-orthonormal, and $V \subseteq \ker L$;*

(B) *$n = \varkappa_V$ and there exist scalar matrices $\{A_{k,j}\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$, $\{B_{k,j}\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$ such that*

(B-i) $X_j Q_k \overset{V}{=} A_{k,j} Q_{k+1} + B_{k,j} Q_k + A_{k-1,j}^* Q_{k-1}$ *for all $j \in \overline{1, N}$ and $k \in \overline{0, \varkappa_V}$,*

(B-ii) $\ell(Q_k) \leqslant d_V(k)$ *for every $k \in \overline{0, \varkappa_V}$,*

(B-iii) $\deg Q_k \leqslant k$ *for every $k \in \overline{0, \varkappa_V}$.*

Under the above assumptions
the conditions (A) and (B) are equivalent.

Notice the absence of the rank condition in part (B) of Theorem 8.

Moreover the following holds:

1° the condition (B) implies the injectivity of $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ for every $k \in \overline{0, \varkappa_V}$ and the positive definiteness of L, where L is given by

$$L|_V = 0,\ L(Q_0) = Q_0,\ L(Q_k) = 0\ \forall\ k \in \overline{1, \varkappa_V}, \tag{3}$$

2° the condition (A) implies $V = \mathcal{V}_L$; if a linear functional $L' : \mathcal{P}_N \to \mathbb{C}$ satisfies (A), then L' fulfils (3).

Orthogonalizing measures:
$\mathfrak{M}_N$ = the set of all positive Borel measures μ on $\mathbb{R}^N$ with all finite moments, i.e.

$$\int_{\mathbb{R}^N} |x^\alpha| \, \mathsf{d}\mu(x) < \infty \quad \text{for all } \alpha \in \mathbb{N}^N.$$

Given $\mu \in \mathfrak{M}_N$, we define the linear functional $L_\mu : \mathcal{P}_N \longrightarrow \mathbb{C}$ via

$$L_\mu(p) = \int_{\mathbb{R}^N} p \, \mathsf{d}\mu, \quad p \in \mathcal{P}_N.$$

If $L = L_\mu$ for some $\mu \in \mathfrak{M}_N$, then L is called a **moment functional (induced by μ)**.

Every moment functional is positive definite but not conversely.

We say that a measure $\mu \in \mathfrak{M}_N$ **orthonormalizes** a sequence $\{Q_k\}_{k=0}^n$ of column polynomials ($0 \leqslant n \leqslant \infty$) if $\{Q_k\}_{k=0}^n$ is L_μ-orthonormal.

Zariski topology:
For $\varDelta \subseteq \mathbb{R}^N$, denote by $\mathcal{I}(\varDelta)$ the $*$-ideal:

$$\mathcal{I}(\varDelta) = \{p \in \mathcal{P}_N : p(x) = 0 \text{ for all } x \in \varDelta\}.$$

Call $\mathcal{I}(\varDelta)$ a **set ideal** (**induced by the set** $\varDelta$). There are $*$-ideals which are not set ideals (e.g. $V = (X^s) \subseteq \mathcal{P}_1$, $s \geqslant 2$).

If $\operatorname{int}(\varDelta) \neq \varnothing$, then $\mathcal{I}(\varDelta) = \{0\}$.

For $p \in \mathcal{P}_N$, denote by $\mathcal{Z}_p$ the **algebraic set:**

$$\mathcal{Z}_p \stackrel{\text{df}}{=} \{x \in \mathbb{R}^N : p(x) = 0\}.$$

Define

$$\overline{\varDelta}^{\mathsf{z}} = \bigcap \Big\{\mathcal{Z}_p : p \in \mathcal{P}_N \text{ and } \varDelta \subseteq \mathcal{Z}_p\Big\}. \qquad (4)$$

The set $\overline{\varDelta}^{\mathsf{z}}$ remains unchanged if we replace $\mathcal{P}_N$ by $\mathcal{R}_N$ in (4).

Since each algebraic subset of $\mathbb{R}^N$ is of the form $\mathcal{Z}_p$ with some $p \in \mathcal{R}_N$, our definition of $\overline{\varDelta}^{\mathsf{z}}$ coincides with the closure of $\varDelta$ in the Zariski topology (which consists of complements of algebraic subsets of $\mathbb{R}^N$).

$\mathcal{V}_{L_\mu}$ is a set ideal:

Recall that

$$\mathcal{V}_L = \bigcap_{q \in \mathcal{P}_N} \{p \in \mathcal{P}_N : L(pq) = 0\},$$
$$\varkappa_V = \dim \mathcal{P}_N / V.$$

Proposition 9 *If* $\mu \in \mathfrak{M}_N$ *and* $\mu \neq 0$, *then*

(i) $\mathcal{V}_{L_\mu} = \mathcal{I}(\operatorname{supp}\mu) = \mathcal{I}(\overline{\operatorname{supp}\mu}^{\mathrm{z}})$,
(ii) $\operatorname{supp}\mu$ *is finite if and only if* $\varkappa_{\mathcal{V}_{L_\mu}} < \infty$ (*equivalently:* $\dim \mathcal{P}_N / \mathcal{V}_{L_\mu} < \infty$).

The condition (B) and set ideals:
The problem of existence of an orthonormalizing measure can only be solved in the case of set ideals.

Proposition 10 *Let V be a proper $*$-ideal in $\mathcal{P}_N$ and $\varnothing \neq \Delta \subseteq \mathbb{R}^N$.*

(i) *If a sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of real column polynomials* (*with* $Q_0 = 1$) *satisfies the condition* (B) *of Theorem* 8, *and L defined by* (2) *is a moment functional induced by $\mu \in \mathfrak{M}_N$, then* $V = \mathcal{I}(\operatorname{supp} \mu)$.

(ii) *If $V = \mathcal{I}(\Delta)$, then there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ composed of real column polynomials* (*with* $Q_0 = 1$) *which is orthonormalized by some $\mu \in \mathfrak{M}_N$ and which satisfies the condition* (B) *of Theorem* 8.

Types A, B and C:

• A polynomial $p \in \mathcal{P}_N$ is of **type A** if every positive definite linear functional $L : \mathcal{P}_N \to \mathbb{C}$ vanishing on the principal ideal $(p) \subseteq \mathcal{P}_N$ is a moment functional.

$\Longleftrightarrow$ for every inner product space $\mathcal{D}$ and for every cyclic commuting N-tuple $S = (S_1, \ldots, S_N)$ of symmetric operators on $\mathcal{D}$ annihilated by p (i.e. $p(S) = 0$), there exists an N-tuple $(T_1, \ldots, T_N)$ of spectrally commuting selfadjoint operators in a Hilbert space $\mathcal{K} \supseteq \mathcal{D}$ (isometric embedding) such that $S_j \subseteq T_j$ for all $j = 1, \ldots, N$.

• A polynomial $p \in \mathcal{P}_N$ is of **type B**, if every positive definite linear functional $L : \mathcal{P}_N \to \mathbb{C}$ vanishing on $\mathcal{I}(\mathcal{Z}_p)$ is a moment functional.

• A $*$-ideal V in $\mathcal{P}_N$ is said to be of **type C**, if either $V = \mathcal{P}_N$ or $V \neq \mathcal{P}_N$ and for every sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of real column polynomials satisfying condition (B) of Theorem 8 with $Q_0 = 1$, the linear functional L defined by (2) is a moment functional.

• A polynomial $p \in \mathcal{P}_N$ is of **type C** if the set ideal $\mathcal{I}(\mathcal{Z}_p)$ is of type C.

Proposition 11 *A $*$-ideal V in $\mathcal{P}_N$ is of type* C *if and only if every positive definite linear functional $L : \mathcal{P}_N \to \mathbb{C}$ satisfying $\mathcal{V}_L = V$ is a moment functional.*
If $p \in \mathcal{P}_N$ is of type A*, then it is of type* B*.*
If $p \in \mathcal{P}_N$ is of type B*, then it is of type* C*.*

Type A depends on the number of indeterminates, e.g.

the zero polynomial is of type A as a member of $\mathcal{P}_1$ and is not of type A as a member of $\mathcal{P}_2$.

Theorem 12 *Every polynomial $p \in \mathcal{P}_N$ with compact $\mathcal{Z}_p$ is of type* A*, and, in consequence, of type* C (*the case $\mathcal{Z}_p = \varnothing$ is not excluded*).

Theorem 13 *Assume that $p \in \mathcal{P}_N$ is of type* A *and Δ is a finite subset of $\mathbb{R}^N$. Then the $*$-ideal $\mathcal{I}(\mathcal{Z}_p \cup \Delta)$ in $\mathcal{P}_N$ is of type* C*. In particular, $*$-ideals $\mathcal{I}(\mathcal{Z}_p)$ and $\mathcal{I}(\Delta)$ in $\mathcal{P}_N$ are of type* C*.*

This not longer true for infinite sets Δ.

Examples:

The polynomial $p = X_1^2 X_2^2 (X_1^2 + X_2^2 - 1) + 1 \in \mathcal{P}_2$ is positive. It is not a sum of squares of real polynomials (this is related to Hilbert's 17th problem). The polynomial p is of type A.

Every nonzero polynomial $p \in \mathcal{R}_2$ of degree at most 2 is of type A (J.S. 1992).

All polynomials $X_2 + q(X_1) \in \mathcal{P}_2$ with $q \in \mathcal{P}_1$ are of type A.

A polynomial $1+(X_1 \pm \mathrm{i}\, X_2) q(X_1, X_2) \in \mathcal{P}_2$ with $q \in \mathcal{P}_2$ is of type A (J.S. and F.H. Szafraniec 1998).

Bisgaard (1998) completely characterized polynomials of the form $X^\alpha - X^\beta \in \mathcal{P}_N$, $\alpha, \beta \in \mathbb{N}^N$, which are of type A.

J.S. and F.H. Szafraniec (2003) gave examples of polynomials of type A including

$$(X_1 + \mathrm{i}\, X_2) q(X_1, X_2) X_3 - 1 \in \mathcal{P}_3,$$

where $q \in \mathcal{P}_2$, and

$$(1 + q_1(X_1)^2 + \ldots + q_k(X_k)^2) r(X_1, \ldots, X_k) X_{k+1} - 1 \in \mathcal{P}_{k+1}$$

where $q_1, \ldots, q_k \in \mathcal{R}_1$ are polynomials of degree at least 1 and $r \in \mathcal{R}_k$.

Despite the above instances, one may still find ideals which are not of type C. Set $N = 2$ and $p = (X_2 - X_1^2)X_2 \in \mathcal{P}_2$. The polynomial p is not of type A (J.S. 1992) and so there exists at least one positive definite linear functional L on $\mathcal{P}_2$ vanishing on (p), which is not a moment functional. Then, by Proposition 11, the ideal $V \stackrel{\text{df}}{=} \mathcal{V}_L$ is not of type C. Moreover, we have

$$\{0\} \subsetneq (p) = \mathcal{I}(\mathcal{Z}_p) \subseteq \mathcal{V}_L \subseteq \ker L \subsetneq \mathcal{P}_2.$$

It is not known whether $\mathcal{V}_L = \mathcal{I}(\mathcal{Z}_p)$ for some such L.

The polynomial $p = (X_2 - X_1^2)X_1 \in \mathcal{P}_2$ is of type A.

Question 1 *Do types A and B (resp. B and C) coincide for every* $p \in \mathcal{P}_N$*?*

Question 2 *Is every nonzero set ideal* V *in* $\mathcal{P}_N$ *of type C?*

Question 3 *Is the zero ideal in* $\mathcal{P}_N$ *of type C?*

An answer in the negative to Question 3 implies an answer in the negative to Question 2 (with a greater N).

Orth. measures: matrix approach

Assume that $\varkappa_V = \infty$, V is a proper $*$-ideal in $\mathcal{P}_N$ and $\{Q_k\}_{k=0}^{\varkappa_V}$ is a sequence of real column polynomials (with $Q_0 = 1$) satisfying the condition (B) of Theorem 8.

Arrange the set $\{q + V : q \in \bigcup_{k=0}^{\infty} Q_k\}$ in an orthonormal basis $\{q_k + V\}_{k=0}^{\infty}$ of the Hilbert space completion of $\mathcal{P}_N/V$ relative to $\langle\cdot,-\rangle_L$:

$$\langle q + V, r + V\rangle_L \overset{\text{df}}{=} L(qr^*), \quad q, r \in \mathcal{P}_N$$

respecting the order of columns as well as the order of entries in each column. Then for every $j = 1, \ldots, N$, the multiplication operator $M_{X_j} \in \boldsymbol{L}_{\mathsf{s}}^{\#}(\mathcal{P}_N/V)$ may be represented by the infinite symmetric matrix

$$\boldsymbol{S}_j \overset{\text{df}}{=} \begin{bmatrix} B_{0,j} & A_{0,j} & 0 & 0 & \\ A_{0,j}^{\mathsf{T}} & B_{1,j} & A_{1,j} & 0 & \ddots \\ 0 & A_{1,j}^{\mathsf{T}} & B_{2,j} & A_{2,j} & \ddots \\ 0 & 0 & A_{2,j}^{\mathsf{T}} & B_{3,j} & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with respect to the orthonormal basis $\{q_k + V\}_{k=0}^{\infty}$, where $A_{k,j}$ and $B_{k,j}$ are the real matrices appearing in (B-i) of Theorem 8.

Orth. measures: matrix approach

Theorem 14 *Let V be a proper $*$-ideal in $\mathcal{P}_N$ with $\varkappa_V = \infty$ and $\{Q_k\}_{k=0}^{\infty}$ be a sequence of real column polynomials* (*with $Q_0 = 1$*) *satisfying the condition* (B) *of Theorem* 8. *Assume that there exists a sequence $\{c_n\}_{n=0}^{\infty}$ of positive real numbers such that $\sum_{n=0}^{\infty} c_n^{-1/2} = \infty$ and for every $n \geqslant 0$,*

$$c_n \geqslant \max\Big\{\Big\| \sum_{j=1}^{N} A_{n-1,j}A_{n,j}\Big\|, \Big\| \sum_{j=1}^{N} A_{n,j}A_{n+1,j}\Big\|, \Big\| \sum_{j=1}^{N} (B_{n,j}A_{n,j} + A_{n,j}B_{n+1,j})\Big\|\Big\}.$$

Then there exists a unique measure $\mu \in \mathfrak{M}_N$ which orthonormalizes $\{Q_k\}_{k=0}^{\infty}$ and which satisfies the equality $\mathcal{I}(\operatorname{supp} \mu) = V$.

Orth. measures: matrix approach

Theorem 15 *Let V be a proper $*$-ideal in $\mathcal{P}_N$ with $\varkappa_V = \infty$ and $\{Q_k\}_{k=0}^{\infty}$ be a sequence of real column polynomials* (*with $Q_0 = 1$*) *satisfying the condition* (B) *of Theorem* 8. *Assume that for any two integers $i, j \in \overline{1, N}$ with $i < j$ there exists a sequence $\{c_n^{i,j}\}_{n=0}^{\infty}$ of positive real numbers such that $\sum_{n=0}^{\infty}(c_n^{i,j})^{-1/2} = \infty$ and $c_n^{i,j} \geqslant \max\{a_{n-1}^{i,j}, a_n^{i,j}, b_n^{i,j}\}$ for every $n \geqslant 0$, where*

$$a_n^{i,j} \stackrel{\text{df}}{=} \|A_{n,i}A_{n+1,i} + A_{n,j}A_{n+1,j}\|,$$

$$b_n^{i,j} \stackrel{\text{df}}{=} \|B_{n,i}A_{n,i} + A_{n,i}B_{n+1,i} + B_{n,j}A_{n,j} + A_{n,j}B_{n+1,j}\|.$$

Then there exists a unique measure $\mu \in \mathfrak{M}_N$ which orthonormalizes $\{Q_k\}_{k=0}^{\infty}$ and which satisfies the equality $\mathcal{I}(\operatorname{supp}\mu) = V$.