Perturbations of Orthogonal Polynomials With Periodic Recursion Coefficients

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OPRL and OPUC

We will mainly consider Jacobi matrices

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots \\ a_1 & b_2 & a_2 & \dots & \dots \\ 0 & a_2 & b_3 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and the associated orthogonal polynomials (OPRL: RL = real line) which obey ($\{a_n, b_n\}_{n=1}^{\infty}$ are called Jacobi parameters)

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_np_{n-1}(x)$$

$d\rho$ is the spectral measure for J and $(1, 0, \dots)^t$. $J \to d\rho$ via spectral theorem. $d\rho \to J$ by OPs and recursion relations.

We will also say something about OPUC (UC = unit circle); $\Phi_n(z)$ = monic OP

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z)$$

$$\Phi_n^*(z) = z^n \, \overline{\Phi_n(1/\bar{z})}$$

$$\{\alpha_n\}_{n=0}^{\infty} \leftrightarrow \mathcal{C} \leftrightarrow d\mu$$

$\mathcal{C}$ = CMV matrix

$|\alpha_n| < 1$ and are called Verblunsky coefficients.

Main Results for (Slow) Decay to Free Case

[1] **Weyl**: $a_n \to 1, b_n \to 0 \Rightarrow \sigma_{\text{ess}}(J) = [-2, 2]$

$\sigma_{\text{ess}} \equiv$ non-isolated points of $d\rho$

[2] **Denisov–Rakhmanov**: $\sigma_{\text{ess}}(J) = \Sigma_{\text{ac}}(J) = [-2, 2]$
$\Rightarrow a_n \to 1, b_n \to 0$

$\Sigma_{\text{ac}} =$ essential support of a.c. spectrum, i.e.,

$$d\rho = f(x)\, dx + d\rho_{\text{s}} \qquad \Sigma_{\text{ac}} = \{x \mid f(x) \neq 0\}$$

3 **Szegő**: Suppose that $\sigma_{\text{ess}}(J) = [-2,2]$; $\sum_j (|E_j| - 2)^{1/2} < \infty$. Then the following are equivalent:

(i) $\inf(a_1 a_2 \ldots a_n) > 0$

(ii) $\sum_{j=1}^{\infty} |a_j - 1|^2 + |b_j|^2 < \infty$; $\sum (a_j - 1)$ and $\sum b_j$ cond. conv.

(iii) $\int_{-2}^{2} \log(f(x))(4 - x^2)^{-1/2}\, dx > -\infty$ (Szegő condition)

4 **Killip-Simon**: $\sum_{j=1}^{\infty} |a_j - 1|^2 + |b_j|^2 < \infty \Leftrightarrow$

(i) $\sigma_{\text{ess}}(J) = [-2,2]$

(ii) $\sum (|E_j| - 2)^{3/2} < \infty$

(iii) $\int_{-2}^{2} \log(f(x))(4 - x^2)^{1/2}\, dx > -\infty$ (Pseudo-Szegő condition)

Hints for Periodic Case

$a_n^{(0)}$, $b_n^{(0)}$ are periodic, i.e., $a_{n+p}^{(0)} = a_n^{(0)}$, $b_{n+p}^{(0)} = b_n^{(0)}$

Weyl still holds: $|a_n - a_n^{(0)}| + |b_n - b_n^{(0)}| \to 0 \Rightarrow \sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(J^{(0)})$

Deift–Killip still holds (Theorem of Killip in his thesis):
$\sum |a_n - a_n^{(0)}|^2 + |b_n - b_n^{(0)}|^2 < \infty \Rightarrow \Sigma_{\text{ac}}(J) = \Sigma_{\text{ac}}(J^{(0)})$

Only one direction.

Special case OPUC $\alpha_n^{(0)} = a$ (Geronimus polynomials). θ_0 given by $\sin(\frac{1}{2}\theta_0) = |a|$. Then $\Sigma_{\text{ac}} = \sigma_{\text{ess}} = \{e^{i\theta} \mid \theta_0 \le \theta \le 2\pi - \theta_0\} \equiv \Sigma_{|a|}$.

Lopez has developed much info about perturbations in this case.

Theorem (Barrios–Lopez). $|\alpha_n| \to |a|$, $(\alpha_{n+1}/\alpha_n) \to 1 \Rightarrow \sigma_{\text{ess}}(\mathcal{C}) = \Sigma_{|a|}$

Theorem (Bello–Lopez; Simon; Alfaro et al.; Barrios et al.). $\sigma_{\text{ess}}(\mathcal{C}) = \Sigma_{\text{ac}}(\mathcal{C}) = \Sigma_{|a|} \Rightarrow |\alpha_n| \to |a|$, $(\alpha_{n+1}/\alpha_n) \to 1$

Isospectral Tori, 1

For each $\lambda \in \partial\mathbb{D}$, $\alpha_n^{(\lambda)} = \lambda|a|$. These are all the periodic α_n's with $\Sigma_{\text{ac}} = \Sigma_{|a|}$. The isospectral set is a circle. Lopez theorems are results about approach to this circle of α's.

Returning to OPRL, recall theory of periodic case. Look at whole line $\{a_n^{(0)}, b_n^{(0)}\}_{n=1}^p$ as parameters. Whole line J commutes with $u_n \to u_{n+p}$ so direct integral decomposition

$$J = \int^{\oplus} J(\theta)\, \frac{d\theta}{2\pi}$$

$J(\theta)$ is Jacobi matrix on $\ell_\theta^\infty = \{u \mid u_{n+p} = e^{i\theta} u_n\}$.

Isospectral Tori and Discriminants

Transfer matrix ($a_0 = a_p$)

$$T(x) = \frac{1}{a_p}\begin{pmatrix} b_p - x & a_{p-1} \\ a_p & 0 \end{pmatrix} \frac{1}{a_{p-1}}\begin{pmatrix} b_{p-1} - x & a_{p-2} \\ a_{p-1} & 0 \end{pmatrix} \cdots \frac{1}{a_1}\begin{pmatrix} b_1 - x & a_0 \\ a_1 & 0 \end{pmatrix}$$

$\det T = 1$. Discriminant defined by

$$\Delta(x) = \mathrm{Tr}(T(x))$$

Polynomial of degree p. Eigenvalues of T are

$$\lambda_\pm = \frac{\Delta(x)}{2} \pm \sqrt{\left(\frac{\Delta(x)}{2}\right)^2 - 1}$$

Bounded solns $\Leftrightarrow |\lambda_\pm| = 1 \Leftrightarrow \Delta(x) \in [-2, 2]$.

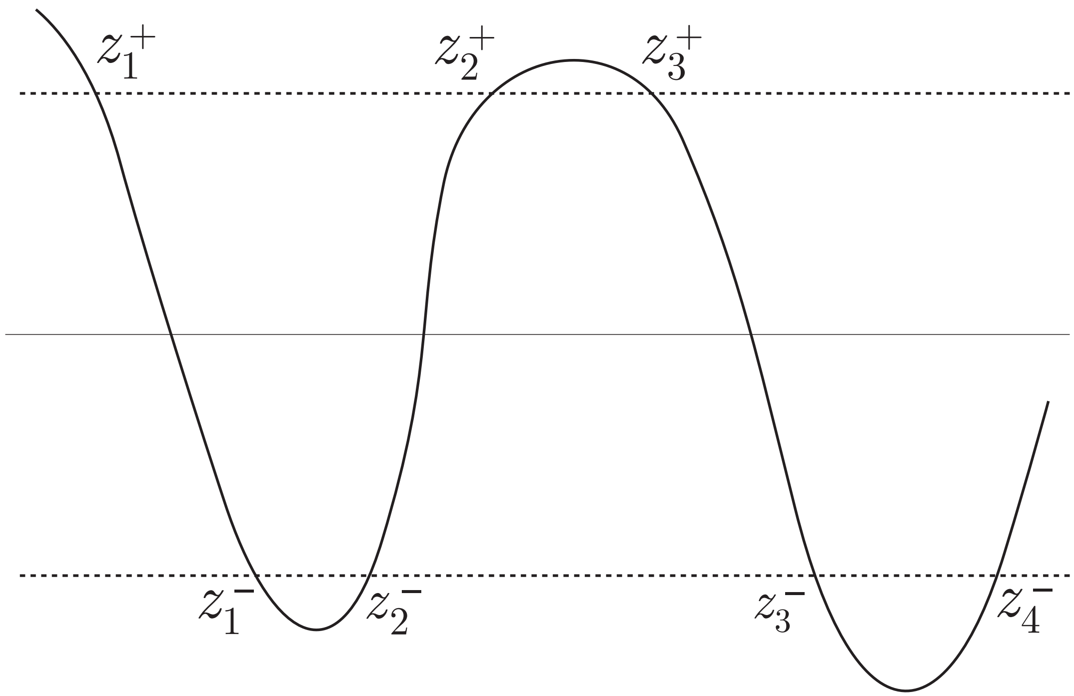
x is eigenvalue of $J(\theta) \Leftrightarrow \lambda_\pm = e^{\pm i\theta} \Leftrightarrow \Delta(x) = 2\cos\theta$

$$\Rightarrow \Delta(J(\theta)) = 2\cos\theta \mathbf{1}$$

($\Delta(x)$ is a polynomial, so $\Delta(\text{op})$ is well-defined.)

$$\sigma(J) = \Delta^{-1}([-2,2])$$

All roots of $\Delta(x) - 2\cos\theta$ have $x \in \mathbb{R}$.



L gaps; generically L = p – 1. Always L ≤ p – 1.

Isospectral Tori, 3

$\{a_n, b_n\}_{n=1}^{p} \to \Delta(x)$ maps $\mathbb{R}^{2p} \to$ polynomials of degree p

Inverse image of point is generically a $p-1$ dim manifold. Tori of dim L.

Conjectures in OPUC Book

Analogs of Big Four but distance to isospectral torus replaces distance to $a_n \equiv 1$, $b_n \equiv 0$. Last–Simon proved Weyl-like theorem. Damanik–Killip–Simon get the others (with extra—all gaps open—assumptions in the Szegő and Killip–Simon analogs).

The Big Theorems

Given two sets of Jacobi parameters:

$$d_n((a,b),(a',b')) \equiv \sum_{m=0}^{\infty} e^{-m}[|a_{n+m} - a'_{n+m}| + |b_{n+m} - b'_{n+m}|]$$

If $\mathcal{T}$ is an isospectral torus,

$$d_n((a,b),\mathcal{T}) \equiv \min(d_n((a,b),(a',b')) \mid (a',b') \in \mathcal{T})$$

J_0 is two-sided periodic. $\mathcal{T}$ is isospectral torus of J_0.

Theorem 1 (Last–Simon). *If* $d_n((a,b),\mathcal{T}) \to 0$, *then* $\sigma_{\text{ess}}(J) = \sigma(J_0)$.

Theorem 2 (DKS). *If* $\sigma_{\text{ess}}(J) = \sigma(J_0)$ *and* $\Sigma_{\text{ac}}(J) = \sigma(J_0)$, *then* $d_n((a,b),\mathcal{T}) \to 0$.

Theorem 3 (DKS). *Suppose* $\sigma_{\text{ess}}(J) = \sigma(J_0)$. $\sum_j \text{dist}(E_j, \sigma(J_0))^{1/2} < \infty$. *Suppose all gaps are open. Then the following are equivalent:*

$$\text{(i)} \qquad \inf(a_1 \dots a_n / a_1^{(0)} \dots a_n^{(0)}) > 0$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} d_n((a,b),\mathcal{T})^2 < \infty$$

$\lim_{N\to\infty} \sum_{m=0}^{N} (a_{mp+1} \dots a_{mp+p}) - (a_1^{(0)} \dots a_p^{(0)})$ *exists and*

$\lim_{N\to\infty} \sum_{m=0}^{N} (b_{mp+1} + \dots + b_{mp+p}) - (b_1^{(0)} + \dots + b_p^{(0)})$ *exists.*

$$\text{(iii)} \qquad \int_{\sigma(J_0)} \log(f(x)) \text{dist}(x, \mathbb{R}\backslash\sigma(J_0))^{-1/2} dx > -\infty$$

Theorem 4 (DKS). *Suppose all gaps are open. Then*

$$\sum_{n=1}^{\infty} d_n((a,b),\mathcal{T})^2 < \infty$$

iff

$$\begin{array}{ll} \text{(i)} & \sigma_{\text{ess}}(J) = \sigma(J_0) \\ \text{(ii)} & \sum_j \text{dist}(E_j, \sigma(J_0))^{3/2} < \infty \\ \text{(iii)} & \int_{\sigma(J_0)} \log(f(x))\text{dist}(x, \mathbb{R}\backslash\sigma(J_0))^{1/2}\, dx < \infty \end{array}$$

For Theorem 1, $\sigma_{\text{ess}}(J) \supset \sigma(J_0)$ is "easy" half and $\sigma_{\text{ess}}(J) \subset \sigma(J_0)$ is "hard" half. The DKS method also provides alternate proof of the hard half of Theorem 1.

I'll sketch the proof of Theorem 2 (Denisov–Rakhmanov analog).

The Magic Formula

Theorem. *J_0 is a two-sided periodic Jacobi matrix with discriminant Δ. J is bounded and two-sided. S is defined by*

$$(Su)_n = u_{n-1}$$

Then

$$\Delta(J) = S^p + S^{-p} \Leftrightarrow J \in \mathcal{T}$$

Aside on OPUC: $\mathcal{C}_0$ is two-sided periodic with even period p. Δ is a Laurent polynomial

$$\Delta(z) = \left(\prod_{j=0}^{p-1} \rho_j\right)^{-1} [z^{p/2} + c_1 z^{p/2-1} + \cdots + z^{-p/2}]$$

Theorem. *$\mathcal{C}_0$ is two-sided periodic CMV matrix with discriminant Δ. $\mathcal{C}$ is two-sided. Then*

$$\Delta(\mathcal{C}) = S^p + S^{-p} \Leftrightarrow \mathcal{C} \in \mathcal{T}$$

This reduces perturbations of periodic CMV to complex matrix-valued Jacobi matrices. (*Note*: Jacobi not CMV!)

Proof of the Magic Formula (OPRL)

One Direction. $J \in \mathcal{T} \Rightarrow \Delta_J = \Delta_{J_0}$ so it suffices to prove $\Delta_{J_0}(J_0) = S^p + S^{-p}$. Passing to direct integrals, this is equivalent to $\Delta(J_0(\theta)) = 2\cos\theta$, which we noted earlier!

Other Direction.

$$\Delta(x) = \frac{1}{a_1^{(0)} \dots a_p^{(0)}} \left[x^p - \left[\sum_{j=1}^{p} b_j^{(0)} \right] x^{p-1} + O(x^{p-2}) \right]$$

$$\Delta(J)_{k,k+p} = \frac{a_k \, a_{k+1} \dots a_{k+p-1}}{a_1^{(0)} \dots a_p^{(0)}}$$

This is $\equiv 1 \Rightarrow a_k \dots a_{k+p-1} = \text{constant} \Rightarrow a_k = a_{k+p}$. Similarly, looking at $\Delta(J)_{k,k+p-1} = 0 \Rightarrow b_k = b_{k+p}$. Periodic, same $\prod_{j=1}^{p} a_j$'s and $\Sigma_{\text{ac}}(J) \subset \Sigma_{\text{ac}}(J_0) \Rightarrow J \in \mathcal{T}$.

Block Jacobi Matrices

$\Delta(J)$ is a block diagonal matrix-valued Jacobi matrix with $p \times p$ blocks. The A_n's are not self-adjoint. Instead they are lower triangular with positive diagonal elements.

Theorem. (Denisov–Rakhmanov for Block Jacobi). *K is $p \times p$ block diagonal with A_n of the above form. Suppose $\sigma_{\text{ess}}(K) = [-2, 2]$ and the $p \times p$ matrix measure $d\rho(x) = M(x)\,dx + d\rho_s$ has* $\det(M) > 0$ *a.e. on* $[-2, 2]$. *Then $B_n \to 0$, $A_n \to \mathbf{1}$.*

This comes from:

1. van Assche proved Rakhmanov for matrix OPUC.

2. Yakhlef–Marcellan used van Assche but with a different normalization from ours (their A_n's are picked so $A_n A_{n-1} \dots A_1$ is symmetric). Their result implies our A_n's obey $A_n^* A_n \to \mathbf{1}$.

3. Lower triangular with positive on diagonal plus $A_n^* A_n \to \mathbf{1}$ implies $A_n \to \mathbf{1}$ (little lemma).

4. Carry Denisov's ideas over to go from van Assche–Yakhlef–Marcellan $\Rightarrow$ Rakhmanov to get full Denisov–Rakhmanov.

Proof of Theorem 2. $K = \Delta(J)$. Since p bands of J map each to $[-2, 2]$, K_{ac} has multiplicity p. Thus, block Jacobi DR applies. This means that any right limit of K is $S^p + S^{-p}$. (Here right limit means strong limit of $S^m K S^{-m}$ on $\ell^2(\mathbb{Z})$). If J_∞ is a right limit of J, $\Delta(J_\infty)$ is thus $S^p + S^{-p}$. Therefore, all right limits of J are in $\mathcal{T}$. Compactness completes the proof.

Remarks on L^2 Results

Where does the all-gaps-open hypothesis come in? In general, one gets a Killip–Simon-type theorem for matrix-valued measures. On one side, one gets $\sum_n \text{Tr}(B_n^* B_n) + \text{Tr}((A_n - 1)^*(A_n - 1)) < \infty$. On the other side, one gets a term like $\int \log(\det f)(4 - x^2)^{1/2}\, dx$. Handling the $\det f$ is a little tricky, but the key difficulty is going from ℓ^2 bounds on distance of $\Delta(J)$ to $S^p + S^{-p}$ to ℓ^2 bounds on $d_n((a,b),\mathcal{T})$.

The key fact that holds when all gaps are open is that the gradients of the coefficients of Δ span the normal bundle to $\mathcal{T}$. That cannot be true if any gaps are closed.

The magic formula has lots of other applications.

A Final Note

From the point of view of recursion coefficients, these are complete results (except for the still missing statements when there are closed gaps). But from the point of view of measures, they are VERY incomplete.

Consider for example Denisov–Rakhmanov. Our result says something about a finite set of intervals. If the a.c. and essential spectrum is that set of intervals, then the recursion coefficients approach the isospectral tori associated with those intervals. BUT one needs the extra hypothesis that the harmonic measures of the intervals are all rational — because only then are we perturbing about a periodic problem.

If the intervals are general, there is still an isospectral torus, but it is of almost periodic potentials. There is an obvious counterpart of Denisov–Rakmanov but we have no idea how to prove it! For almost periodic problems have no discriminant and so no magic formula.