Equilateral quantum graphs

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Motivation

Quantum graphs are “metrizations” of combinatorial graphs. Can one relate the spectral properties of a quantum graph with its combinatorial counterpart?

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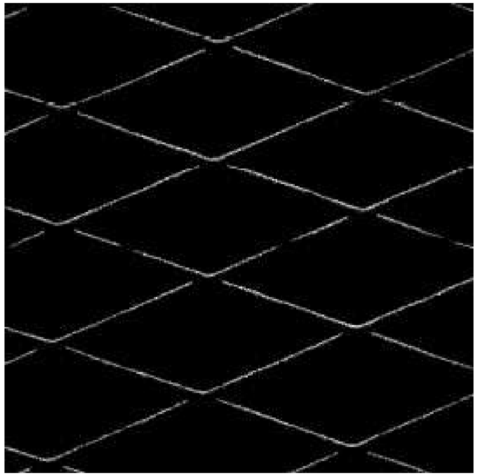
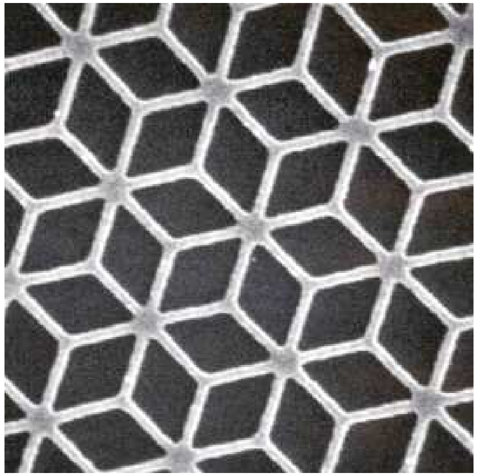
General equilateral structures with potentials?

Superconducting networks



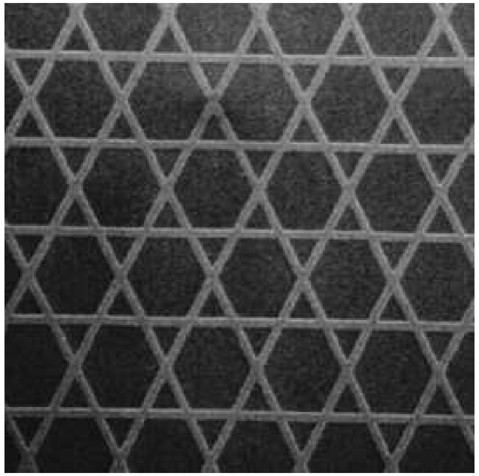
Square aluminium wire network. The wires are 80 nm thick, 250 nm wide. The wire length is 3 μm

B. Pannetier et al., Surface Science **229** (1990) 331



Silver T_3 wire network. The wire width is 50 nm, the length is 0.4μm

C. Naud, Ann. Phys. (Paris) **27**:2 (2002)



Aluminium kagomé network. The wires are 250 nm wide and 2 μm long

M. J. Higgins et al., Phys. Rev. B **61** (2000) R894

Combinatorial graph

Let G be a directed graph, V the set of vertices, E the set of edges (both countable), self-loops and multiple edges allowed.

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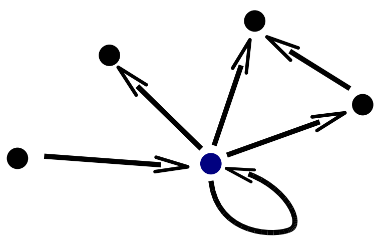

Blue vertex: $\operatorname{indeg} v = 2$, $\operatorname{outdeg} v = 4$, $\deg v = 6$

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Discrete Hilbert space $l^2(G)$: $f : V \to \mathbb{C}$ summable with respect to the scalar product

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Discrete Laplacian (tight-binding Hamiltonian)

$$\Delta f(v) = \frac{1}{\deg v}\Big(\sum_{e:\iota e = v} e^{-i\beta(e)} f(\tau e) + \sum_{e:\tau e = v} e^{i\beta(e)} f(\iota e).\Big)$$

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Δ is self-adjoint and bounded in $l^2(G)$, $\|\Delta\| \leq 1$.

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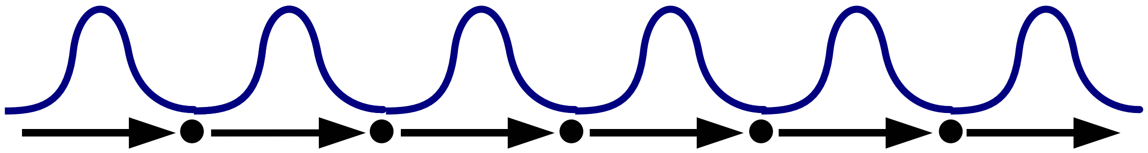
Magnetic Kirchhoff boundary condition:

$$f_b(0) = f_e(1) \text{ for } \iota b = \tau e,$$

$$\sum_{e:\iota e = v} \left(\frac{d}{dt} - i a_e\right) f_e(0) - \sum_{e:\tau e = v} \left(\frac{d}{dt} - i a_e\right) f_e(1) = 0 \text{ for all } v.$$

Example: Hill operator

Extend V by periodicity and consider $P = -\frac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$: a particular case of the above construction, the graph G is the simple chain



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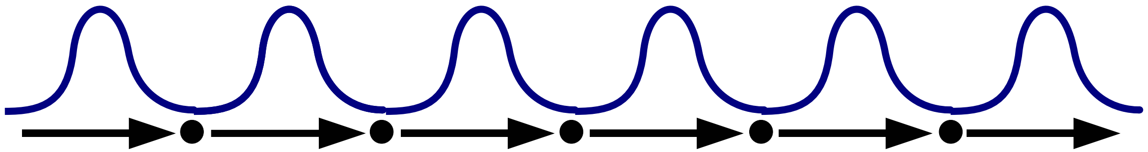

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Let $s(\cdot,z), c(\cdot,z)$ be the fundamental solutions to

$$-u'' + Vu = zu, \quad s(0,z) = c'(0,z) = 0, \quad s'(0,z) = c(0,z) = 1.$$

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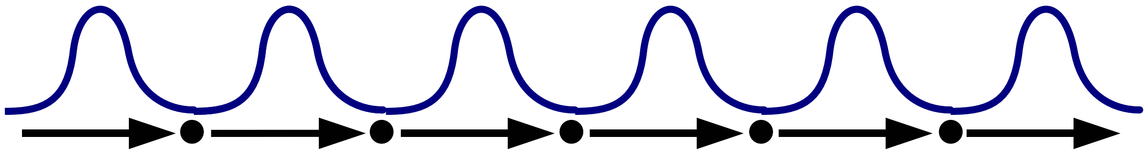

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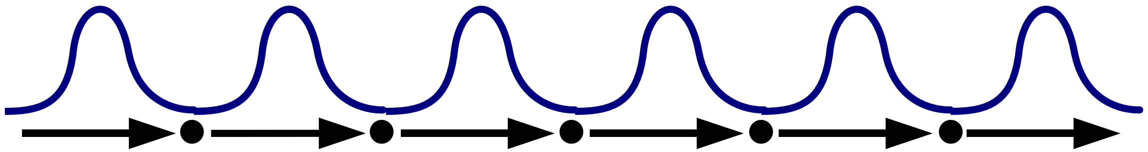

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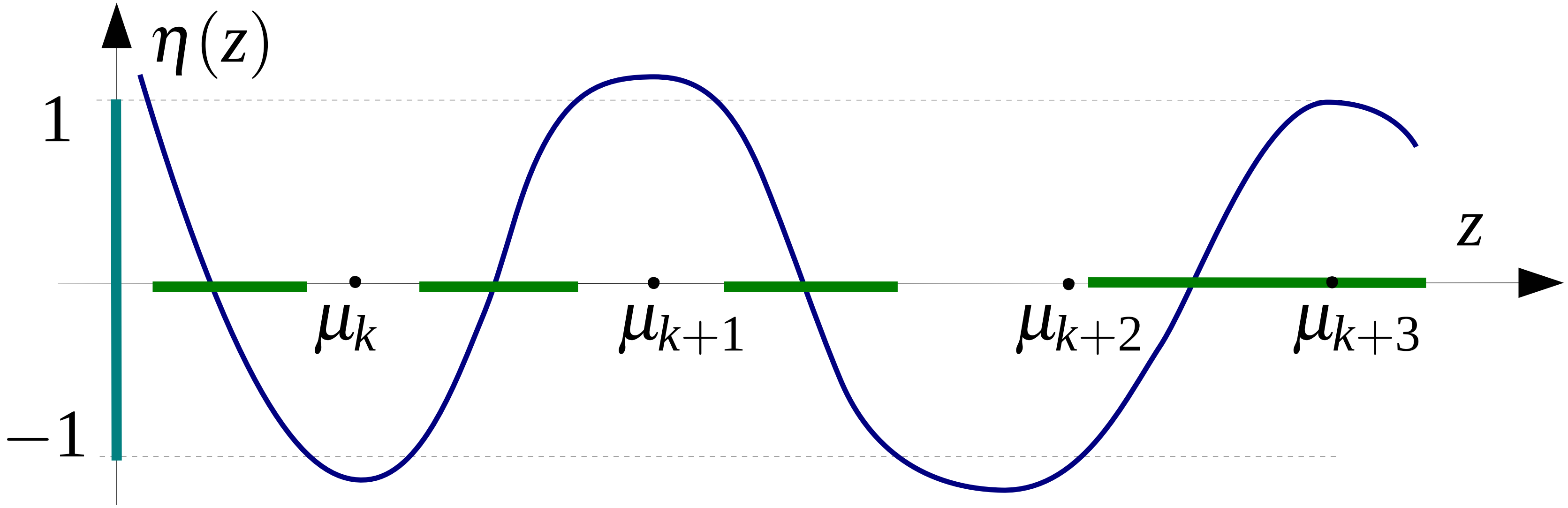
The spectrum $\operatorname{spec} P = \eta^{-1}([-1,1])$ is absolutely continuous

Spectrum of Hill operator



- μ_k are the Dirichlet eigenvalues of $-\dfrac{d^2}{dx^2} + V$ on $[0, 1]$,
- $\left|\eta(\mu_k)\right| \geq 1$,
- $\eta'(z) \neq 0$ for $\eta(z) \in (-1, 1)$,
- the gaps (if open) are situated near μ_k.

Main theorem

Let one of the following conditions be satisfied:

$$(a)\ \operatorname{indeg} v = \operatorname{outdeg} v \text{ for all } v, \quad (b)\ V(x) \equiv V(1-x),$$

then $\operatorname{spec} L = \Sigma_0 \cup \Sigma$, where $\Sigma_0 \subset \{\mu_k\}$ (a discrete set) and $\Sigma = \eta^{-1}(\operatorname{spec} \Delta)$, where Δ is the discrete magnetic Laplacian with $\beta(e) = \int_0^1 a_e(t)\, dt$ and η is the Hill discriminant for V.

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Moreover, outside $\{\mu_k\}$ one has $\operatorname{spec}_j L = \eta^{-1}(\operatorname{spec}_j \Delta)$ for $j \in \{p, pp, disc, ess, ac, sc, c, sing\}$. Therefore, only Δ is responsible for the quality of the spectrum.

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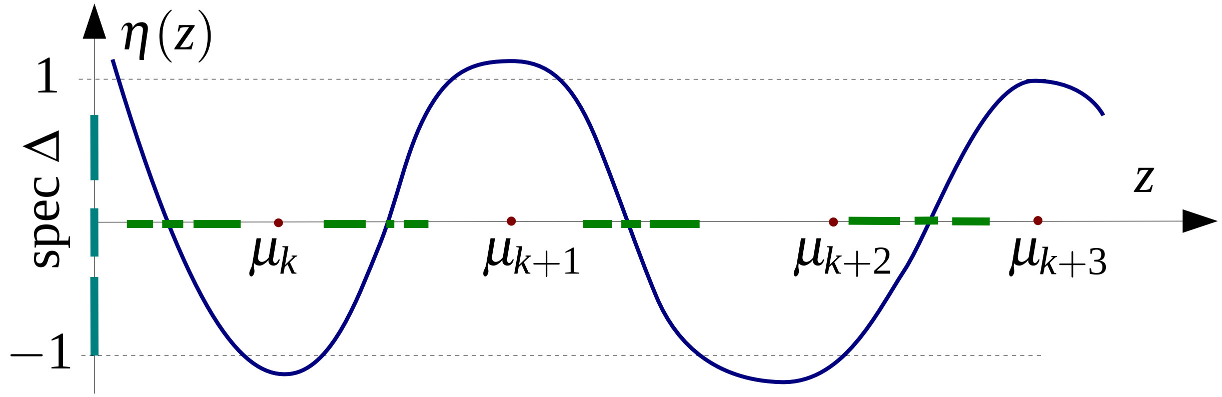
Technique: the discrete Laplacian with some coefficients is exactly the Weyl function in suitable coordinates + Krein’ s resolvent formula!

Spectrum of quantum graph



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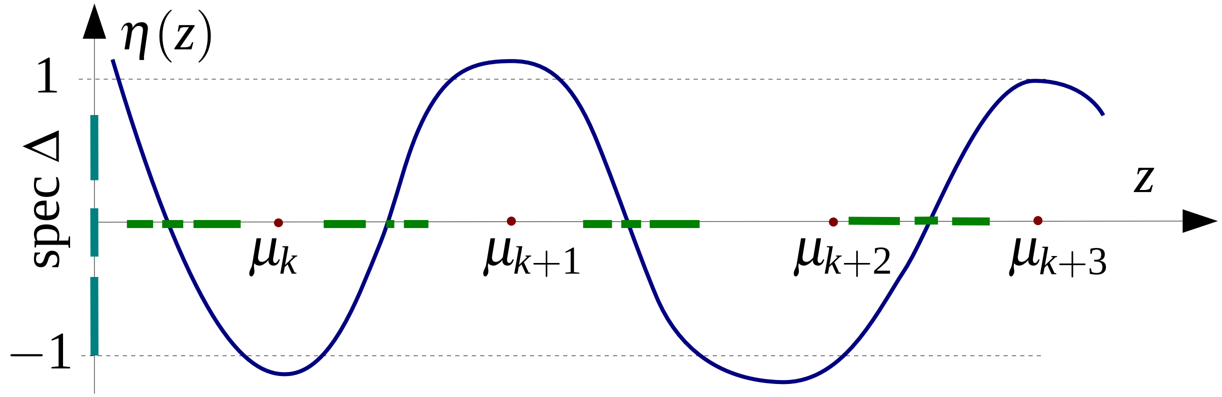

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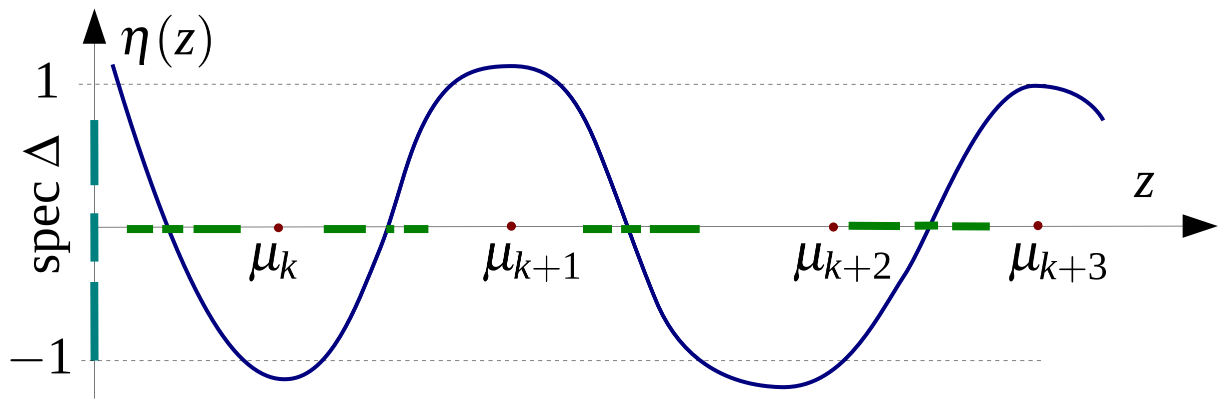

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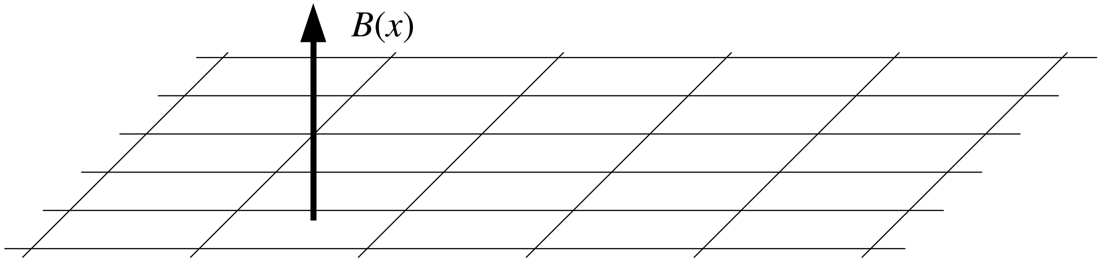
Corollaries:

- the gap of P near μ_k is open $\Rightarrow L$ has a gap near μ_k,
- $\operatorname{spec}\Delta \neq [-1,1] \Rightarrow L$ has infinitely many gaps,
- L has finitely many gaps $\Leftrightarrow V$ is finite-gap and $\operatorname{spec}\Delta = [-1,1]$...

Example: Cantor spectrum

Square lattice in orthogonal magnetic field, $\theta := \int_{\text{cell}} \frac{B(x)}{2\pi}\, dx$ constant



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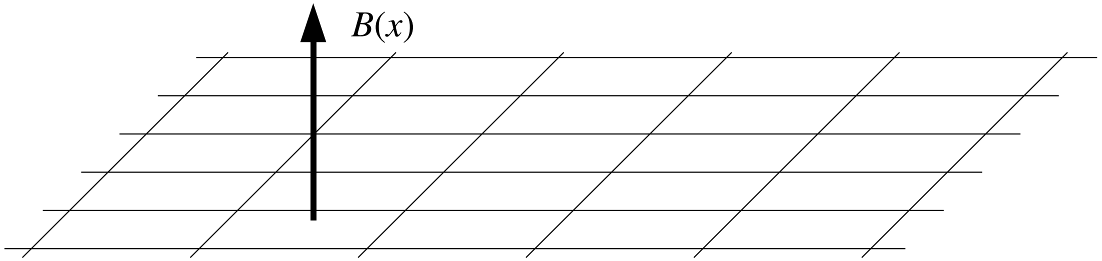

The discrete Laplacian is the Harper operator in $l^2(\mathbb{Z}^2)$:

$$\Delta f(m,n) = \frac{1}{4}\big(e^{i\pi\theta n} f(m-1,n) + e^{-i\pi\theta n} f(m,n+1) \\ + e^{-i\pi\theta m} f(m,n-1) + e^{i\pi\theta m} f(m,n+1)\big).$$

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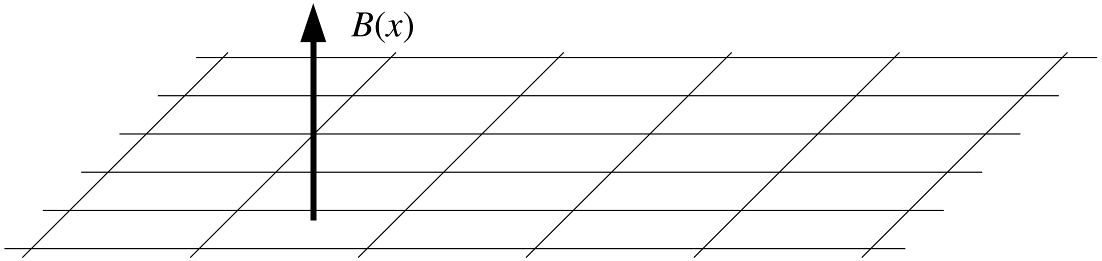

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For irrational θ the spectrum of Δ is a Cantor set (the ten martini problem, Avila & Jitomirskaya'05), hence also the spectrum of L. The Dirichlet eigenvalues are infinitely degenerate.

Possible extensions (1)

- Uniformly directed graphs: $\dfrac{\operatorname{indeg} v}{\deg v} =: \alpha$ is constant.

  Up to a discrete set:

  $$\operatorname{spec}\Lambda = \eta^{-1}(\operatorname{spec}\Delta), \quad \eta(z) = \alpha s'(1;z) + (1-\alpha)c(1;z).$$

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- Bipartite graphs: $V = V_1 \cup V_2$, edges only between V_1, V_2,

  $$\alpha_j = \frac{\operatorname{indeg} v_j}{\deg v_j} \text{ constant for } v_j \in V_j.$$

  One has up to a discrete set $\operatorname{spec}\Lambda = \eta^{-1}(\operatorname{spec}\Delta^2)$,

  $$\eta(z) = \big(\alpha_1 s'(1;z) + (1-\alpha_1)c(1;z)\big)\big(\alpha_2 s'(1;z) + (1-\alpha_2)c(1;z)\big).$$

Possible extensions (2)

- More general boundary conditions. For example, δ-type couplings lead to $\Sigma = \eta_\alpha^{-1}(\operatorname{spec}\Delta)$, where η_α is the Hilll discriminant for the Kronig-Penney operator

$$-\frac{d^2}{dx^2} + V + \alpha \sum_{n\in\mathbb{Z}} \delta(\cdot - n).$$

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- For generic boundary conditions one arrives at matrix discrete Laplacians.

References

- K. Pankrashkin, *Spectra of Schrödinger operators on equilateral quantum graphs*, Lett. Math. Phys. (to appear)
- J. Brüning, V. Geyler, K. Pankrashkin, *Cantor and band spectra for periodic quantum graphs with magnetic fields*, Commun. Math. Phys. (to appear)
- + papers in preparation

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