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Lax-Phillips scattering revisited

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1 Pseudo-Hamiltonians

Open quantum systems are often described by maximal dissipative operators which are called pseudo-Hamiltonians.

The closed linear operator H in $\mathfrak{H}$ is called dissipative if

$$\Im m(Hf, f) \leq 0, \quad f \in \mathrm{dom}(H).$$

Advantage: There is always a larger Hilbert space $\mathfrak{L}$ containing $\mathfrak{H}$ as subspace and a self-adjoint operator K_H in $\mathfrak{L}$ such that

$$P^{\mathfrak{L}}_{\mathfrak{H}}(K_H - z)^{-1} \upharpoonright \mathfrak{H} = (H - z)^{-1}, \quad z \in \mathbb{C}_+.$$

The operator K_H is called a self-dilation of H. The dilation is called minimal if

$$\mathfrak{L} = \mathrm{clospan}\{(K_H - \lambda)^{-1}\mathfrak{H} : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$$

2 Quasi-Hamiltonians

The minimal self-adjoint dilation K_H of H is regarded as the minimal closed system $\{K_H, \mathfrak{L}\}$ in which the open system $\{H, \mathfrak{H}\}$ is embedded and is called the quasi-Hamiltonian referring to H.

In the following we always assume that H is a maximal dissipative extension of a closed symmetric operator A with finite deficiency indices, that is, $A \subseteq H$.

One chooses $\mathfrak{G} = L^2(\mathbb{R}, dx, \mathfrak{g})$, $\dim(\mathfrak{g}) < \infty$, and introduces the symmetric operator G

$$(Gf)(x) = -i\frac{d}{dx}f(x), \quad f \in \mathrm{dom}(G) := \{f \in W^{1,2}(\mathbb{R}, \mathfrak{g}) : f(0) = 0\}. \tag{1}$$

in $L^2(\mathbb{R}, dx, \mathfrak{g})$. It turns out that

$$\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{G}$$

and

$$A \oplus G \subseteq K_H.$$

3 Scattering

We consider the self-adjoint extension K_0 of $A \oplus G$,

$$K_0 = A_0 \oplus G_0$$

where A_0 is a self-adjoint extension of A and G_0 is a self-adjoint extension of G given by

$$(G_0 f)(x) = -i\frac{d}{dx} f(x), \quad f \in \mathrm{dom}(G_0) := W^{1,2}(\mathbb{R}, \mathfrak{g}).$$

The wave operators

$$W_\pm(K_H, K_0) := \text{s-}\lim_{t\to\pm\infty} e^{itK_H} e^{-itK_0} P^{ac}(K_0)$$

exist are complete, that means, $\{K_H, K_0\}$ performs a complete scattering system.

$$S_H := W_+(K_H, K_0)^* W_-(K_H, K_0) : \mathfrak{L}^{ac}(K_0) \longrightarrow \mathfrak{L}^{ac}(K_0)$$

is called the scattering operator of $\{K_H, K_0\}$ which is unitary on the absolutely continuous subspace $\mathfrak{L}^{ac}(K_0)$.

4 Partial wave operators

$$W_\pm(K_H, A_0) := \text{s-}\lim_{t\to\pm\infty} e^{itK_H} e^{-itA_0} P^{ac}(A_0).$$

$$W_\pm(K_H, G_0) := \text{s-}\lim_{t\to\pm\infty} e^{itK_H} e^{-itG_0}, \quad P^{ac}(G_0) = I_{\mathfrak{G}}.$$

Notice that

$$\mathrm{ran}(W_\pm(K_H, A_0)) \perp \mathrm{ran}(W_\pm(K_H, G_0))$$

and

$$\mathrm{ran}(W_\pm(K_H, A_0)) \oplus \mathrm{ran}(W_\pm(K_H, G_0)) = \mathfrak{L}^{ac}(K_H).$$

Hence

$$S_H := \begin{pmatrix} S(A_0, A_0) & S(A_0, G_0) \\ S(G_0, A_0) & S(G_0, G_0) \end{pmatrix},$$

where

$$S(A_0, A_0) = W_+(K_H, A_0)^* W_-(K_H, A_0), \quad S(A_0, G_0) = W_+(K_H, A_0)^* W_-(K_H, G_0)$$
$$S(G_0, A_0) = W_+(K_H, G_0)^* W_-(K_H, A_0), \quad S(K_0, K_0) = W_+(K_H, G_0)^* W_-(K_H, G_0)$$

5 Scattering matrix

In the spectral representation $L^2(\mathbb{R}, d\lambda, \mathcal{L}_\lambda)$ of K_0^{ac} the scattering matrix is unitary equivalent to the multiplication operator induced by the scattering matrix $\{S_H(\lambda)\}_{\lambda\in\mathbb{R}}$. The values $S_H(\lambda)$ are unitary operators.

Let $L^2(\mathbb{R}, d\lambda, \mathcal{H}_\lambda)$ and $L^2(\mathbb{R}, d\lambda, \mathfrak{g})$ be the spectral representation of A_0^{ac} and $G_0 = G_0^{ac}$, respectively. Then

$$L^2(\mathbb{R}, d\lambda, \mathcal{L}_\lambda) = L^2(\mathbb{R}, d\lambda, \mathcal{H}_\lambda \oplus \mathfrak{g}), \quad \mathcal{L}_\lambda = \mathcal{H}_\lambda \oplus \mathfrak{g},$$

and the scattering matrix $S_H(\lambda)$ decomposes into

$$S_H(\lambda) = \begin{pmatrix} S(A_0, A_0)(\lambda) & S(A_0, G_0)(\lambda) \\ S(G_0, A_0)(\lambda) & S(G_0, G_0)(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

6 Lax-Phillips scattering

$\{K_H, K_0\}$ performs a Lax-Phillips scattering system:

$$\mathcal{D}_\pm := L^2(\mathbb{R}_\pm, dx, \mathfrak{g})$$

are the incoming and outgoing subspace, that means,

$$e^{-itK_H}\mathcal{D}_\pm \subseteq \mathcal{D}_\pm, \quad t \in \mathbb{R}_\pm.$$

The Lax-Phillips wave operator are identical with the partial wave operators $W_\pm(K_H, G_0)$. Consequently, $S(G_0, G_0)$ is the Lax-Phillips scattering operator and $\{S(G_0, G_0)(\lambda)\}_{\lambda\in\mathbb{R}}$ is the Lax-Phillips scattering matrix.

From Adamyan and Arov one knows that

$$S(G_0, G_0)(\lambda) = W_H(\lambda - i0)^*, \quad \lambda \in \mathbb{R},$$

where $W_H(z)$, $z \in \mathbb{C}_-$, is the characteristic function of the maximal dissipative operator H.

7 Inner and outer systems

- $\{A_0, \mathfrak{H}\}$ and $\{G_0, \mathfrak{G}\}$ are called inner and outer system, respectively.
- $\{K_H, \mathfrak{L}\}$ is an interacting system while $\{K_0 = A_0 \oplus G_0, \mathfrak{L}\}$ is non-interacting.
- $\{K_H, \mathfrak{L}\}$ is a singular perturbation of $\{K_0, \mathfrak{L}\}$.
- The interaction is hidden in the pseudo-Hamiltonian H.
- The disadvantage is that the outer world is too simple. Indeed, the Hamiltonian of the outer world is up to the multiplicity always the same: the momentum operator.
- The Hamiltonian K_H is always not semibounded from below.

8 How to improve the model?

Let us replace the outer system $\{G_0, \mathfrak{G}\}$ by another one $\{T_0, \mathfrak{K}\}$ where

$$T_0 \supseteq T$$

of a densely defined closed symmetric operator T with finite deficiency indices in $\mathfrak{K}$.
Let us choose an extension

$$L \supseteq A \oplus T$$

such that

$$P^{\mathfrak{L}}_{\mathfrak{H}}(L - z)^{-1} \restriction \mathfrak{H} = (H(z) - z)^{1}, \quad z \in \mathbb{C}_+,$$

where $\{H(z)\}_{z\in\mathbb{C}_+}$ is a family of maximal dissipative extensions of A.

$\{L_0 = A_0 \oplus T_0, \mathfrak{L}\}$ is the non-interacting system, $\{L, \mathfrak{L}\}$ is the interacting system and $\{H(z)\}_{z\in\mathbb{C}_+}$ is called the Strauss family.

$$T_0 \longleftrightarrow G_0 \qquad L \longleftrightarrow K_H \qquad H(z) \longleftrightarrow H.$$

9 Coupling of symmetric operators: Existence

Let

$$n_-(A) = n_+(A) = n_-(T) = n_+(T) < \infty.$$

Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $\{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$ for T^*. Then $\{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$,

$$\widetilde{\mathcal{H}} = \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{H} \end{matrix} \qquad \widetilde{\Gamma}_0 = \begin{pmatrix} \Gamma_0 \\ \Upsilon_0 \end{pmatrix}, \qquad \widetilde{\Gamma}_1 = \begin{pmatrix} \Gamma_1 \\ \Upsilon_1 \end{pmatrix},$$

is a boundary triplet for $A^* \oplus T^*$. Choosing the self-adjoint relation

$$\Theta = \left\{ \begin{pmatrix} \begin{pmatrix} v \\ v \end{pmatrix} \\ \begin{pmatrix} w \\ -w \end{pmatrix} \end{pmatrix} : v, w \in \mathcal{H} \right\}$$

one gets an extension $L := L_\Theta$ which obeys the previous conditions (Derkach, Hassi, Malamud, de Snoo 2004).

10 Strauss family

In the framework of boundary triplets the Strauss family $\{H(z)\}_{z\in\mathbb{R}}$ is given by

$$H(z) := A^* \restriction \ker(\Gamma_1 + \tau(z)\Gamma_0), \quad z \in \mathbb{C}_+,$$

where $\tau(z)$ is Weyl function of the boundary triplet $\{\mathcal{H}, \Upsilon_0, \Upsilon_1\}$.

$$P^{\mathfrak{L}}_{\mathfrak{H}}(L - z)^{-1} \restriction \mathfrak{H} = (H(z) - z)^{-1}, \quad z \in \mathbb{C}_+.$$

L can be called a “generalized dilation” of the Strauss family. In abstract operator theory L is called a linearization. The generalized dilation L is called minimal

$$\mathfrak{L} = \{(L - z)^{-1}\mathfrak{H} : z \in \mathbb{C} \setminus \mathbb{R}\}.$$

The minimal generalized dilations of a Strauss family are unitarily equivalent.

A Nevanlinna function $\tau(z)$ is a Weyl function of some boundary triplet if and only if

$$\lim_{y\to+\infty} \frac{(\tau(iy)h, h)}{y} = 0, \quad h \in \mathcal{H}, \quad \text{and} \quad \lim_{y\to\infty} (\tau(iy)h, h) = \infty \quad 0 \neq h \in \mathcal{H}.$$

11 Scattering system

The unperturbed Hamiltonian is given by $L_0 = A_0 \oplus T_0$,

$$A_0 := A^* \restriction \ker(\Gamma_0) \qquad \text{and} \qquad K_0 := T^* \restriction \ker(\Upsilon_0).$$

The scattering system $\{L, L_0\}$. The scattering operator $S = W_+(L, L_0)^* W_-(L, L_0)$ admits the decomposition

$$S = \begin{pmatrix} S(A_0, A_0) & S(A_0, T_0) \\ S(T_0, A_0) & S(T_0, T_0) \end{pmatrix}$$

The scattering matrix admits in the spectral representation $L^2(\mathbb{R}, d\lambda, \mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)})$, $\mathcal{H}_{M(\lambda)} = \mathrm{ran}(\Im\mathrm{m}(M(\lambda + i0)))$, $\mathcal{H}_{\tau(\lambda)} = \mathrm{ran}(\Im\mathrm{m}(\tau(\lambda + i0)))$, the representation

$$S(\lambda) = I_{\mathcal{H}_{M(\lambda)} \oplus \mathcal{H}_{\tau(\lambda)}} - 2i\sqrt{\Im\mathrm{m}(\widetilde{M}(\lambda))} \begin{pmatrix} (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \\ (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \end{pmatrix} \sqrt{\Im\mathrm{m}(\widetilde{M}(\lambda))}$$

where

$$\widetilde{M}(\lambda) := \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}.$$

12 Coupling and scattering

The Strauss family $H(z)$ admits an extension to the real axis $H(\lambda)$, $\lambda \in \mathbb{R}$, which can be regarded as a pseudo-Hamiltonian at energy λ.

Let us consider the scattering system $\{K_{H(\lambda)}, K_0\}$.

Let $\{S_{H(\lambda)}(\mu)\}_{\mu\in\mathbb{R}}$ be the scattering matrix of $\{K_{H(\lambda)}, K_0\}$.

Then one has that

$$S(\lambda) = S_{H(\lambda)}(\lambda).$$

That means, the scattering matrix of the coupled system can be reconstructed by a suitable posed Lax-Phillips scattering system at each energy λ induced by the Strauss family.

13 Remarks

- $\{A_0, \mathfrak{H}\}$ and $\{T_0, \mathfrak{K}\}$ can be called inner and outer system, respectively.
- $\{L, \mathfrak{L}\}$ is an interacting system while $\{L_0 = A_0 \oplus T_0, \mathfrak{L}\}$ is non-interacting.
- $\{L, \mathfrak{L}\}$ is a singular perturbation of $\{K_0, \mathfrak{L}\}$.
- The interaction is hidden in the Strauss family $\{H(z)\}_{z \in \mathbb{C}_+}$.
- The outer world $\{T_0, \mathfrak{K}\}$ can be arbitrary complicated.
- The Hamiltonian L can be now semi-bounded.
- The new model replaces the old one, however, it does not replace it completely since at each energy it preserves it.

14 Example

14.1 Inner system

$\mathfrak{H} = L^2(x_l, x_r)$,

$$(Af)(x) := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f(x) + V(x)f(x),$$

$$\mathrm{dom}(A) := \left\{ f \in \mathfrak{H} : \begin{array}{c} f, \frac{1}{m}f' \in W_2^1((x_l, x_r)) \\ f(x_l) = f(x_r) = 0 \\ \left(\frac{1}{m}f'\right)(x_l) = \left(\frac{1}{m}f'\right)(x_r) = 0 \end{array} \right\}.$$

$$\Gamma_0 f := \begin{pmatrix} f(x_l) \\ f(x_r) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f := \frac{1}{2}\begin{pmatrix} \left(\frac{1}{m}f'\right)(x_l) \\ -\left(\frac{1}{m}f'\right)(x_r) \end{pmatrix},$$

The boundary triplet is given by $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$. A_0 corresponds to Dirichlet boundary conditions.

14.2 Outer system

$$\mathfrak{H} = L^2(\mathbb{R} \setminus (x_l, x_r)) = L^2(-\infty, x_l)) \oplus L^2(x - r, \infty)),$$

$$(Tg)(x) := \begin{pmatrix} -\frac{1}{2}\frac{d}{dx}\frac{1}{m_l}\frac{d}{dx}g_l(x) + v_l g_l(x) & 0 \\ 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m_r}\frac{d}{dx}g_r(x) + v_r g_r(x) \end{pmatrix},$$

$$\mathrm{dom}(T) := \left\{ g \in \mathfrak{K} : \begin{array}{c} g \in W_2^2((-\infty, x_l)) \oplus W_2^2((x_r, \infty)) \\ g_l(x_l) = g_r(x_r) = g_l'(x_l) = g_r'(x_l) = 0 \end{array} \right\},$$

$$\Upsilon_0 g := \begin{pmatrix} g_l(x_l) \\ g_r(x_r) \end{pmatrix} \quad \text{and} \quad \Upsilon_1 g := \frac{1}{2}\begin{pmatrix} -\frac{1}{m_l}g_l'(x_l) \\ \frac{1}{m_r}g_r'(x_r) \end{pmatrix}, \tag{2}$$

The boundary triplet is given by $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$. T_0 corresponds to Dirichlet boundary conditions.

$$\lambda \mapsto \tau(\lambda) = \begin{pmatrix} i\sqrt{\frac{\lambda - v_l}{2m_l}} & 0 \\ 0 & i\sqrt{\frac{\lambda - v_r}{2m_r}} \end{pmatrix}.$$

14.3 Coupling

$$L(g_l \oplus f \oplus g_r) = \begin{pmatrix} -\frac{1}{2}\frac{d}{dx}\frac{1}{m_l}\frac{d}{dx}g_l + v_l g_l & 0 & 0 \\ 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}f + vf & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d}{dx}\frac{1}{m_r}\frac{d}{dx}g_r + v_r g_r \end{pmatrix}.$$

with

$$g_l(x_l) = f(x_l), \quad f(x_r) = g_r(x_r), \quad \frac{1}{m_l}g_l'(x_l) = \left(\frac{1}{m}f'\right)(x_l), \quad \left(\frac{1}{m}f'\right)(x_r) = \frac{1}{m_r}g_r'(x_r)$$

which is identical with the Buslaev-Fomin operator

$$L = -\frac{1}{2}\frac{d}{dx}\frac{1}{M}\frac{d}{dx} + V,$$

$$V(x) := \begin{cases} v_l & x \in (-\infty, x_l] \\ v(x) & x \in (x_l, x_r) \\ v_r & x \in [x_r, \infty) \end{cases}, \qquad M(x) := \begin{cases} m_l & x \in (-\infty, x_l] \\ m(x) & x \in (x_l, x_r) \\ m_r & x \in [x_r, \infty) \end{cases}.$$

14.4 Strauss family

$$(A_{D(\lambda)}f)(x) := \left(-\frac{1}{2}\frac{d}{dx}\frac{1}{m}\frac{d}{dx}f(x) + V(x)f(x)\right),$$

$$\mathrm{dom}(A_{D(\lambda)}) = \left\{ f \in \mathfrak{H} : \begin{array}{c} f, \frac{1}{m}f' \in W^1_2((x_l, x_r)), \\ \left(\frac{1}{2m}f'\right)(x_l) = -i\sqrt{\frac{\lambda - v_l}{2m_l}}f(x_l), \\ \left(\frac{1}{2m}f'\right)(x_r) = i\sqrt{\frac{\lambda - v_r}{2m_r}}f(x_r) \end{array} \right\}. \tag{3}$$

14.5 Characteristic function

$$W_{D(\lambda)}(\mu) = I_{\mathcal{H}_{\tau(\lambda)}} + 2i\sqrt{\Im\mathrm{m}(\tau(\lambda))}\big(\tau(\lambda)^* + M(\mu)\big)^{-1}\sqrt{\Im\mathrm{m}(\tau(\lambda))}, \tag{4}$$

$\mu \in \mathbb{C}_-$ where $\mathcal{H}_{\tau(\lambda)} = \mathbb{C}^2, \mathbb{C}, \{0\}$.

14.6 Scattering matrix

$$S(\lambda) = W_{D(\lambda)}(\lambda - i0)^*, \quad \lambda \in (\min\{v_l, v_r\}, \infty),$$