Optimal Lieb-Thirring constants for an exactly solvable magnetic Schrödinger operator

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Contents of this talk

- Magnetic Schrödinger operators $(i\nabla + \vec{A})^2 + V$
- The Aharonov-Bohm magnetic field $\vec{A} = (-x_2, x_1)/|x|^2$
- Exact solutions
- The Lieb-Thirring inequality. Optimal constants for $\gamma = 0$ and 1
- What happens when $0 < \gamma < 1$?

A large part of the results have been published in
H., On the spectrum and eigenfunctions of the Schrödinger operator with Aharonov-Bohm magnetic field, *Int. J. Math. Math. Sci.*, **23** (2005) 3751–3766

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Magnetic Schrödinger operators

Let $\vec{B} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a magnetic field and $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ an electrostatic potential. Let $\Omega \subset \mathbb{R}^d$, where $d = 2$ or 3, and define on $L^2(\Omega)$ the operator

$$H = (i\nabla + \vec{A})^2 + V,$$

where the vector potential (gauge) $\vec{A} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is such that $\mathrm{curl}\,\vec{A} = \vec{B}$.

By H we mean the Friedrichs extension, *i.e.*, the closure of $C_0^\infty(\Omega)$ with respect to the graph norm $\sqrt{||u||^2 + a[u]}$, where

$$a[u] = \int_{\mathbb{R}^d} \left(|(i\nabla + \vec{A})u|^2 + V|u|^2 \right) \mathrm{d}x.$$

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Gauge invariance: The vector potential $\vec{A}$ is not unique since $\mathrm{curl}(\vec{A} + \nabla\phi) = \mathrm{curl}\,\vec{A}$ for any scalar function ϕ. Hence,

$$\vec{A}_1 - \vec{A}_2 = \nabla\phi \quad \Longrightarrow \quad e^{i\phi} H(\vec{A}_1) e^{-i\phi} = H(\vec{A}_2).$$

However, the spectrum of $H(\vec{A})$ depends only on $\vec{B}$.

In **one dimension** the unitary transformation

$$(Uf)(x) = e^{-i\int_0^x A(s)\mathrm{d}s} f(x)$$

eliminates any vector potential. $\Rightarrow$ No magnetic fields exist.

The Aharonov-Bohm magnetic field

The AB effect was predicted in 1949 by W. Ehrenberg and R.E. Siday.
It was rediscovered and described in Y. Aharonov & D. Bohm, *Phys. Rev.* **115** (1959)

Physical model:

- infinitely long solenoid along the x_3 axis with magnetic flux $2\pi\alpha$ inside
- the radius is made infinitely small with constant flux
- a *quantum* particle interacts with the resulting field

The motion in the x_3 direction is classical. Take

$$\vec{A}(x_1, x_2) = \alpha\frac{(-x_2, x_1)}{|x|^2} \quad \Longrightarrow \quad \mathrm{curl}\,\vec{A} = (0,0),\ x \neq (0,0).$$

This is a δ-type interaction.

If we add a **constant magnetic field** $\vec{B} = B\vec{e}_3$ we get

$$\vec{A}(x_1, x_2) = \alpha\frac{(-x_2, x_1)}{|x|^2} + \frac{B}{2}(-x_2, x_1).$$

In cylindrical coordinates, $H\big|_{L_z=m} = H_m$, where

$$H_m = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\alpha - m)^2 - \frac{1}{4}}{r^2} + B(\alpha - m) + \frac{B^2}{4}r^2 + V.$$

The field can be 'gauged away' if α is integer, so that we may assume $0 < \alpha < 1$.

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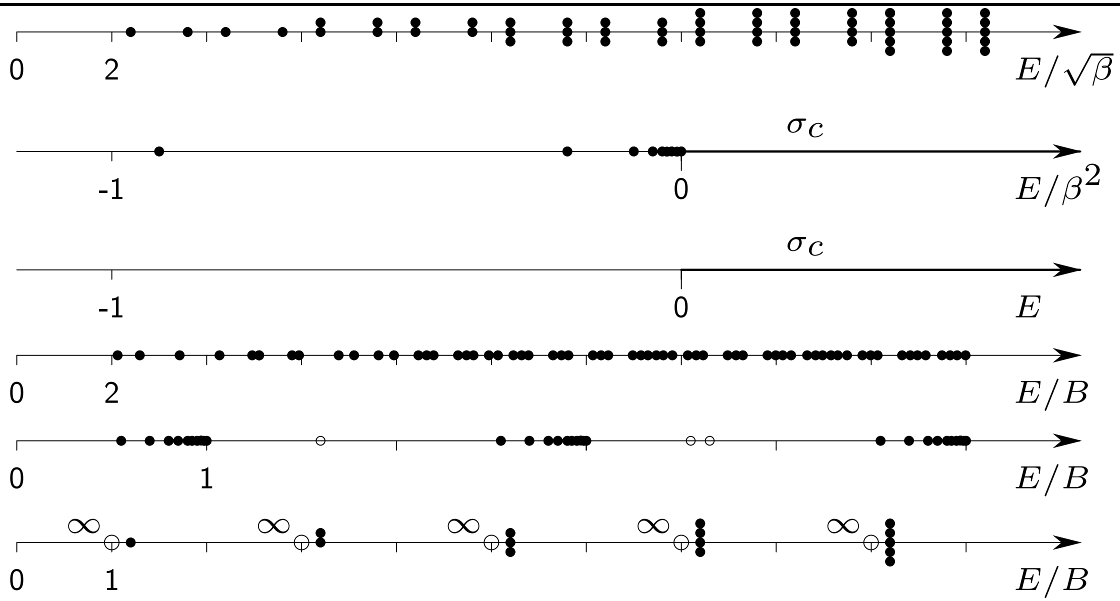
When are exact solutions possible?

The choice of V is rather restricted: $-|x|^{-2}$, $-|x|^{-1}$ or $|x|^2$.

	α	B	V
1.	$*$	0	βr^2
2.	$*$	0	$-\beta r^{-1}$
3.	$*$	0	$-\beta r^{-2}$
4.	$*$	$*$	βr^2
5.	$*$	$*$	$-\beta r^{-2}$
6.	$*$	$*$	0



4–6 are joint work with Rupert Frank

Exact solutions – the details

In **Case 3** H_m is simply the (spherical) Bessel operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu^2 - 1/4}{r^2}$$

with $\nu = \sqrt{(\alpha - m)^2 - \beta}$. Eigenfunctions: $\sqrt{r}J_\nu(\sqrt{E}r),\ \sqrt{r}Y_\nu(\sqrt{E}r)$.

In **all other cases** the eigenvalue problem can be reduced to the confluent hypergeometric equation,

$$-u'' + \left(\frac{\mu^2 - 1/4}{r^2} - \frac{\lambda}{r} + \frac{1}{4}\right) u = 0,$$

whose two solutions are (if $2\mu \notin \mathbb{Z} \setminus \{0\}$) the Whittaker functions

$$M_{\lambda,\pm\mu}(z) = z^{\pm\mu+\frac{1}{2}} e^{-z/2} {}_1F_1(\pm\mu - \lambda + 1/2, 2\mu + 1; z),$$

where ${}_1F_1$ is a hypergeometric series given by

$${}_1F_1(\gamma, \delta; z) = 1 + \frac{\gamma}{\delta}\frac{z}{1!} + \frac{\gamma(\gamma+1)}{\delta(\delta+1)}\frac{z^2}{2!} + \frac{\gamma(\gamma+1)(\gamma+2)}{\delta(\delta+1)(\delta+2)}\frac{z^3}{3!} + \ldots.$$

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How should λ, μ, z be chosen (as functions of $\alpha, \beta, B, E, m, r$) in order that

$$(H_m - E)u = 0 \quad \Longleftrightarrow \quad -u'' + \left(\frac{\mu^2 - 1/4}{z^2} - \frac{\lambda}{z} + \frac{1}{4}\right) u = 0\ ?$$

	H_m	λ_m	μ_m	z
1.	$-\frac{d^2}{dr^2} + \frac{(\alpha-m)^2-1/4}{r^2} + \beta r^2$	$\frac{E}{4\sqrt{\beta}}$	$\frac{1}{2}\lvert\alpha - m\rvert$	$\sqrt{\beta} r^2$
2.	$-\frac{d^2}{dr^2} + \frac{(\alpha-m)^2-1/4}{r^2} - \beta r^{-1}$	$\frac{\beta}{2\sqrt{\lvert E\rvert}}$	$\lvert\alpha - m\rvert$	$2\sqrt{\lvert E\rvert} r$
4.	$-\frac{d^2}{dr^2} + \frac{(\alpha-m)^2-1/4}{r^2} + B(\alpha - m) + \frac{B^2 r^2}{4} + \beta r^2$	$\frac{E-B(\alpha-m)}{2\sqrt{B^2+4\beta}}$	$\frac{1}{2}\lvert\alpha - m\rvert$	$\frac{\sqrt{B^2+4\beta}}{2} r^2$
5.	$-\frac{d^2}{dr^2} + \frac{(\alpha-m)^2-1/4}{r^2} + B(\alpha - m) + \frac{B^2 r^2}{4} - \beta r^{-2}$	$\frac{E-B(\alpha-m)}{2B}$	$\frac{\sqrt{(\alpha-m)^2-\beta}}{2}$	$\frac{B}{2} r^2$
6.	$-\frac{d^2}{dr^2} + \frac{(\alpha-m)^2-1/4}{r^2} + B(\alpha - m)$	$\frac{E-B(\alpha-m)}{2B}$	$\frac{1}{2}\lvert\alpha - m\rvert$	$\frac{B}{2} r^2$

E.g., in Case 2, the (generalised) eigenfunctions are

$$M_{\frac{\beta}{2\sqrt{|E|}},|\alpha-m|}\left(2\sqrt{|E|}r\right)\frac{e^{im\theta}}{\sqrt{r}} = \frac{n!(2\sqrt{|E|}r)^{|\alpha-m|}}{(2|\alpha-m|+1)_n} L_n^{2|\alpha-m|+1}\left(2\sqrt{|E|}r\right) e^{im\theta-\sqrt{|E|}r},$$

where $n = 0, 1, 2, \ldots$ and L_n^a is the Laguerre function.

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The Lieb-Thirring inequality

$$\mathrm{Tr}(-\Delta + V - \lambda)_-^\gamma \le \frac{R_{\gamma,n}}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (a(x,k) - \lambda)_-^\gamma \, \mathrm{d}x \mathrm{d}k, \qquad (*)$$

where

$$a(x,k) = |k|^2 + V(x)$$

is the symbol ($\sim$ classical Hamiltonian) of the operator.

If we carry out the integration w.r.t. k and assume $\sigma(-\Delta + V) \cap (-\infty, \lambda] = \{E_1, E_2, \ldots\}$, we obtain the equivalent inequality

$$\sum_j (E_j - \lambda)_-^\gamma \le R_{\gamma,n} \underbrace{\frac{\Gamma(\gamma+1)}{2^n \pi^{n/2} \Gamma(\gamma + \frac{n}{2} + 1)}}_{= L^{\mathrm{cl}}_{\gamma,n}} \int_{\mathbb{R}^n} (V(x) - \lambda)_-^{\gamma + \frac{n}{2}} \mathrm{d}x.$$

Lieb & Thirring (1976). Review articles: Laptev & Weidl (2000), Blanchard & Stubbe (1996)

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The LT inequality $(*)$ is a semiclassical approximation justified by a principle in Classical Mechanics: The trajectories $(k(t), x(t))$ of each eigenvalue occupies a volume of unit measure in phase space, $\mathbb{R}^n \times \Omega$.

Application 1: Sobolev inequality for N fermions – a lower bound on the kinetic energy

Application 2: Stability of matter – particle systems will not collapse in the absence of exterior forces:

$$\left\langle \vec{x}_j^2 \right\rangle^{1/2} \geq cN^{1/3}, \quad j = 1, \ldots, N.$$

The proof uses Application 1 and the Virial Theorem.

Some known facts about **the constant** $R_{n,\gamma}$:

- $R_{n,\gamma} \geq 1$
- $R_{n,\gamma'} \leq R_{n,\gamma}$ if $\gamma' \geq \gamma$ (Aizenman & Lieb, 1978)
- $R_{2,\gamma} < \infty$ for all $\gamma > \max(0, 1 - n/2)$ (Lieb & Thirring, 1976)

Note that $(*)$ is called Cwikel-Lieb-Rozenblyum's inequality in the special case $\gamma = 0$.

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We now specialise inequality $(*)$ to 2D and the symbol of H,

$$a(x,k) = \left(-\xi_1 - \frac{\alpha x_2}{|x|^2}, -\xi_2 + \frac{\alpha x_1}{|x|^2}\right)^2 + \frac{B^2}{4}|x|^2 + B(\alpha - m) + V(x).$$

The RHS becomes

$$\begin{aligned}
&\frac{R_{2,\gamma}}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(a(x,k)-\lambda)_-^\gamma \mathrm{d}x\mathrm{d}k \\
=&\frac{R_{2,\gamma}}{2(\gamma+1)}\int_{\mathbb{R}^2}\left(\frac{B^2}{4}|x|^2 + V(x) + B(\alpha - m) - \lambda\right)_-^{\gamma+1}\mathrm{d}x \\
=&R_{2,\gamma}\times\begin{cases}\dfrac{(\lambda - B(\alpha-m))^{\gamma+2}}{2\sqrt{B^2+4\beta}(\gamma+1)(\gamma+2)} & \text{if } V(x) = \beta|x|^2; \\ \left(\dfrac{\beta}{2}\right)^2\dfrac{\gamma\pi}{\sin\gamma\pi}|\lambda|^{\gamma-1} & \text{if } V(x) = -\beta|x|^{-1},\ B=0,\ \gamma<1.\end{cases}
\end{aligned}$$

With our complete knowledge of the spectrum it is now straightforward to compute the LHS, $\sum_j (E_j - \lambda)_-^\gamma$, and determine $R_{2,\gamma}$.

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Optimal LT constants for $B = 0$, $\gamma = 0$

It is known that $R_{2,0}$ is infinite for all non-magnetic Schrödinger operators.

Maximising LHS/RHS w.r.t. λ we find

– for $V = \beta|x|^2$:

$$R_{2,0} = \begin{cases} \dfrac{2}{(1+|\alpha - m_1|)^2} & \text{if} \quad 0 < |\alpha - m_1| \le 3\sqrt{2} - 4 \\ \dfrac{1}{(1-\frac{1}{2}|\alpha - m_1|)^2} & \text{if} \quad 3\sqrt{2} - 4 \le |\alpha - m_1| \le \frac{1}{2} \end{cases}$$

– for $V = -\beta|x|^{-1}$:

$$R_{2,0} = \begin{cases} \dfrac{1}{(\frac{1}{2}+|\alpha - m_1|)^2} & \text{if} \quad 0 < |\alpha - m_1| \le 2\sqrt{2} - \frac{5}{2} \\ \dfrac{2}{(\frac{3}{2}-|\alpha - m_1|)^2} & \text{if} \quad 2\sqrt{2} - \frac{5}{2} \le |\alpha - m_1| \le \frac{1}{2} \end{cases}$$

m_1 denotes the best integer approximation of α, hence $0 < |\alpha - m_1| < 1/2$.

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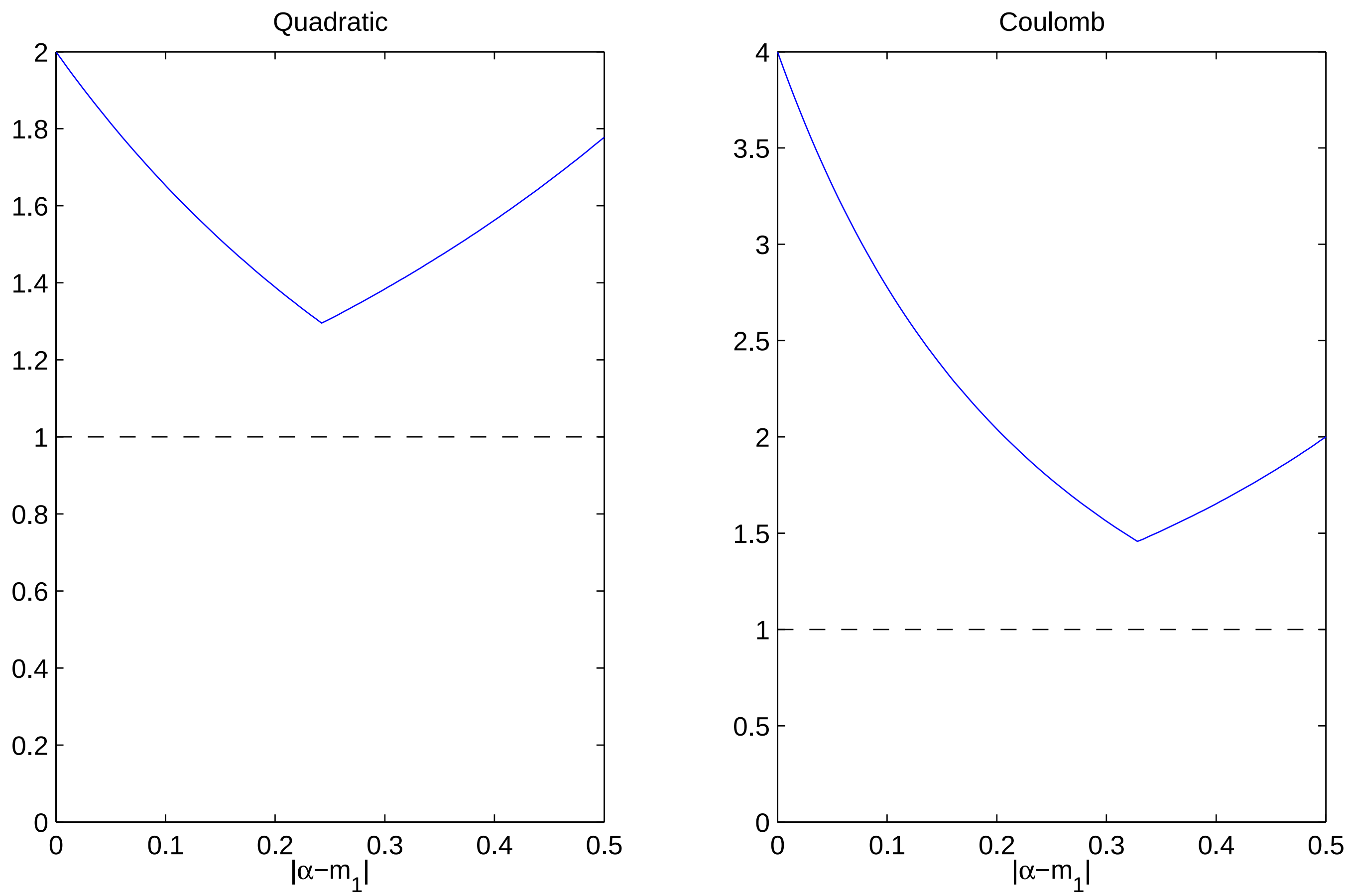
The constants $R_{2,0}$ are not classical for any α.



Optimal LT constants for $B = 0$, $\gamma = 1$

R. de la Bretèche (*Ann. Inst. H. Poincaré Phys. Théor.*, 1999) has shown that the constant is 'classical', *i.e.*,

$$R_{2,\gamma} = 1, \qquad \text{for all } \gamma \geq 1,$$

for the **non-magnetic** harmonic oscillator, $-\Delta + |x|^2$. His proof is a direct calculation.

In this case too,

$$R_{2,1} = 1.$$

By the Aizenman-Lieb principle this implies

$$R_{2,\gamma} = 1 \qquad \text{for all } \gamma \geq 1.$$

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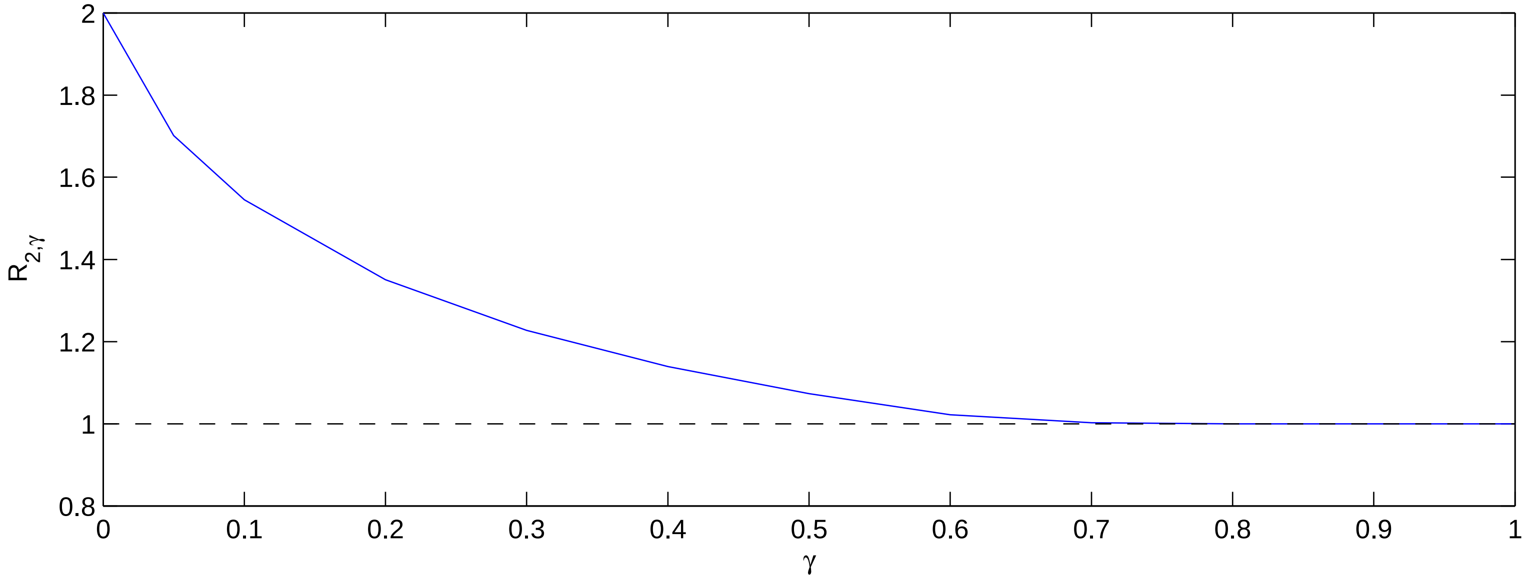
Optimal constants in $0 < \gamma < 1$

If $B = 0$, $V(x) = \beta|x|^2$ we have seen that

$$1.295 < R_{2,0}(\alpha) \leq 2 \qquad \text{but} \qquad R_{2,1} = 1.$$

Is there a $\gamma_c < 1$ for which $R_{2,\gamma_c} = 1$? A numerical study shows that



As expected, $R_{2,\gamma}(\alpha) \leq \lim_{\alpha \downarrow 0} R_{2,\gamma}(\alpha)$ for all γ.

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Thank you for your attention!