Some Applications of (Modified) Fredholm Determinants, or,

Variations on a Theme of Jost and Pais

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OTAMP, Lund, June 15–22, 2006

Topics involved:

- Infinite determinants
- Examples: Jost Functions, Floquet Discriminants, and Evans Functions as Infinite Determinants
- Semiseparable Integral Kernels, Reduction of Infinite Determinants to Finite Ones
- Perturbation Determinants for Schrödinger Operators in Dimensions $n = 1, 2, 3$
- Connections with Dirichlet–Neumann maps

Based on the following work:

- F.G., and K. A. Makarov, *(Modified) Fredholm determinants for operators with matrix-valued semi-separable integral kernels revisited*, Integr. Equ. Operator Theory **47**, 457–497 (2003).
- F.G., Y. Latushkin, M. Mitrea, and M. Zinchenko, *Non-self-adjoint operators, infinite determinants, and some applications*, Russ. J. Math. Phys. **12**, 443–471 (2005).
- F.G., Y. Latushkin, and K. A. Makarov, *Evans functions, Jost functions, and Fredholm determinants*, submitted.
- F.G., M. Mitrea, and M. Zinchenko, *Variations of a Theme of Jost and Pais*, in preparation.

Notation:

Let $\mathcal{H}$ denote a separable complex Hilbert space with the scalar product $(\cdot,\cdot)_{\mathcal{H}}$, and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

$\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$ will denote the Banach spaces of bounded and compact linear operators in $\mathcal{H}$.

Similarly, the Schatten–von Neumann (trace) ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$, $p > 0$ ($\mathcal{B}_1$ denotes the trace class and $\mathcal{B}_2$ the Hilbert–Schmidt class).

Analogous notation $\mathcal{B}(\mathcal{H},\mathcal{K})$, $\mathcal{B}_\infty(\mathcal{H},\mathcal{K})$, etc., will be used for bounded, compact, etc., operators between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

$\overline{T}$ denotes the operator closure of T.

$\mathrm{tr}(T)$ denotes the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$ and $\det_p(I_{\mathcal{H}} - S)$ represents the (modified) Fredholm determinant for $S \in \mathcal{B}_p(\mathcal{H})$, $p \in \mathbb{N}$. (For $p = 1$ we omit the subscript 1 and simply write $\det(I_{\mathcal{H}} - S)$.):

$$T \in \mathcal{B}_1(\mathcal{H}) \text{ then } \det(I_{\mathcal{H}} - T) = \prod_{j \in J} [1 - \lambda_j(T)],$$

where $J \subseteq \mathbb{N}$ (J is finite or infinite), $\lambda_j(T)$ are the (nonzero) eigenvalues of T, and we count them according to their algebraic multiplicity.

Similarly,

$$\begin{aligned} S \in \mathcal{B}_2(\mathcal{H}) \text{ then } \det{}_2(I_{\mathcal{H}} - S) &= \det\big((I_{\mathcal{H}} - S)e^{S}\big) \\ &= \prod_{j \in J} \big[(1 - \lambda_j(S))e^{\lambda_j(S)}\big]. \end{aligned}$$

(The factor $e^{\lambda_j(S)}$ produces convergence of the infinite product.)

Trace Ideals: $\mathcal{B}_p(\mathcal{H})$, $p \geq 1$:

$T \in \mathcal{B}_\infty(\mathcal{H})$, i.e., let T be compact, consider $T^*T \geq 0$, introduce $|T| = (T^*T)^{1/2} \geq 0$, let $s_j(T) \geq 0$, $j \in \mathbb{N}$, be the eigenvalues of $|T|$. The $s_j(T)$ are called the singular values of T.

Then $T \in \mathcal{B}_p(\mathcal{H})$, $p \geq 1$, if $\sum_{j\in\mathbb{N}} s_j(T)^p < \infty$.

$\mathcal{B}_p(\mathcal{H})$, $p \geq 1$ are Banach spaces, $p = 2$ yields the Hilbert space of Hilbert–Schmidt operators, $p = 1$ represents the Banach space of trace class operators,

$$\|T\|_{\mathcal{B}_p(\mathcal{H})} = \left[\sum_{j\in\mathbb{N}} s_j(T)^p\right]^{1/p}, \quad p \geq 1.$$

Examples: Jost Fcts., Floquet Discriminants, and Evans Fcts. as Fredholm Determinants

Example 1. *Jost Functions: R. Jost and A. Pais 1951(half-line case).*

Real line case: $V \in L^1(\mathbb{R}; dx)$, *introduce Jost solutions* $f_\pm(z,\cdot)$ *of the Schrödinger equation* $-\psi''(z) + V\psi(z) = z\psi(z)$, $z \in \mathbb{C}\backslash\{0\}$, *by*

$$f_\pm(z,x) = e^{\pm i z^{1/2} x} - \int_x^{\pm\infty} dx' \, \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} V(x') f_\pm(z,x'),$$

$$\operatorname{Im}(z^{1/2}) \geq 0, \; z \neq 0, \; x \in \mathbb{R},$$

Jost function $\mathcal{F}$*:*

$$\mathcal{F}(z) = \frac{W(f_-(z), f_+(z))}{2iz^{1/2}}$$

$$= 1 - \frac{1}{2iz^{1/2}} \int_{\mathbb{R}} dx \, e^{\mp i z^{1/2} x} V(x) f_\pm(z,x), \quad \operatorname{Im}(z^{1/2}) \geq 0, \; z \neq 0,$$

where $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ *is the Wronskian of* f *and* g.

Consider $H_0 = -d^2/dx^2$ *on* $\mathrm{dom}(H_0) = H^2(\mathbb{R})$ *in* $L^2(\mathbb{R}; dx)$ *and introduce the following factorization of* V,

$$V(x) = u(x)v(x), \quad u(x) = |V(x)|^{1/2}\exp(i\arg(V(x))),\ v(x) = |V(x)|^{1/2}.$$

Consider the operator $K(z)$ *in* $L^2(\mathbb{R}; dx)$ *by*

$$K(z) = -\overline{u\big(H_0 - zI\big)^{-1}v}, \quad z \in \mathbb{C}\backslash\sigma\big(H_0\big).$$

Then, $K(z) \in \mathcal{B}_1(L^2(\mathbb{R}; dx))$, $z \in \mathbb{C}\backslash\sigma\big(H_0\big)$ *and*

$$\det(I - K(z)) = \mathcal{F}(z) = \frac{W(f_-(z), f_+(z))}{2iz^{1/2}}, \quad z \in \mathbb{C},\ \mathrm{Im}(z^{1/2}) > 0.$$

Comments: (i) This is a spectacular reduction: An infinite determinant, $\det(\cdot)$, is reduced to a simple Wronski determinant, $W(\cdot,\cdot)$.

(ii) Eigenvalue eq. $H\psi = E\psi$, $H = H_0 + V$, is related to $\mathcal{F}(E) = 0$.

Example 2. *Floquet Discriminants: Periodic Schrödinger operators of period* $\omega > 0$, *assume* $V \in L^1((0,\omega);dx)$, *and introduce the one-parameter family of self-adjoint operators* $H_{0,\theta}$ *in* $L^2((0,\omega);dx)$ *by*

$$H_{0,\theta} = -d^2/dx^2, \quad \operatorname{dom}\big(H_{0,\theta}\big) = \{g \in L^2((0,\omega);dx) \,|\, g, g' \in AC([0,\omega]); \\ g(\omega) = e^{i\theta} g(0),\, g'(\omega) = e^{i\theta} g'(0),\, g'' \in L^2((0,\omega);dx)\},$$

where $\theta \in [0, 2\pi)$. (*Periodic b.c.'s:* $\theta = 0$, *antiperiodic b.c.'s:* $\theta = \pi$.)

Fundamental system of solutions: $c(z,\cdot)$ *and* $s(z,\cdot)$ *of*
$-\psi''(z) + V\psi(z) = z\psi(z)$,

$$c(z,0) = 1 = s'(z,0), \quad c'(z,0) = 0 = s(z,0), \quad z \in \mathbb{C}.$$

Fundamental matrix of solutions $\Phi(z,x)$:

$$\Phi(z,x) = \begin{pmatrix} c(z,x) & s(z,x) \\ c'(z,x) & s'(z,x) \end{pmatrix}, \quad z \in \mathbb{C}.$$

Monodromy matrix: $\Phi(z,\omega)$.

Floquet discriminant $\Delta(z)$*:*

$$\Delta(z) = \mathrm{tr}_{\mathbb{C}^2}(\Phi(z,\omega))/2 = [c(z,\omega) + s'(z,\omega)]/2, \quad z \in \mathbb{C}.$$

The eigenvalue eq. for $H_\theta = H_{0,\theta} + V$ *then reads,* $\Delta(z) = \cos(\theta)$*.*

Factorize $V = uv$*, and define the operator* $K_\theta(z)$ *in* $L^2((0,\omega);dx)$ *by*

$$K_\theta(z) = -\overline{u\big(H_{0,\theta} - zI\big)^{-1}}v, \quad z \in \mathbb{C}\backslash\sigma\big(H_{0,\theta}\big) \tag{1}$$

Let $\theta \in [0,2\pi)$*, and* $z \in \mathbb{C}\backslash\sigma\big(H_\theta^{(0)}\big)$*. Then,* $K_\theta(z) \in \mathcal{B}_1(L^2((0,\omega);dx))$ *and*

$$\det(I - K_\theta(z)) = \frac{\Delta(z) - \cos(\theta)}{\cos(z^{1/2}\omega) - \cos(\theta)}. \tag{2}$$

Comment: Now an infinite determinant, $\det(\cdot)$, is essentially reduced to a simple trace of the monodromy matrix $\Phi(z,\omega)$ and hence to the Floquet discriminant $\Delta(z)$.

Example 3. *Evans Functions: Consider the "unperturbed" and "perturbed" first-order systems* ($y \in \mathbb{R}^d$, $d \in \mathbb{N}$)

$$y'(x) = A(x)y(x), \quad x \in \mathbb{R},$$
$$y'(x) = (A(x) + R(x))y(x), \quad x \in \mathbb{R},$$

where $A \in L^1_{\rm loc}(\mathbb{R})^{d\times d}$ *and* $\|R\|_{\mathbb{C}^{d\times d}} \in L^1(\mathbb{R}; e^{\beta|x|}\,dx)$ *for some sufficiently large* $\beta > 0$ (*to be determined by certain Bohl and Lyapunov exponents*).

The Evans Function, a $d \times d$*-dimensional determinant, and a tool in linear stability theory for nonlinear evolution equations, turns out to be a Jost-type function and equal to a modified* Fredholm *determinant* (*up to an exponential factor*).

Comments: (*i*) *Again an infinite determinant*, $\det_2(\cdot)$, *is reduced to a* $d \times d$ *determinant*, $\det_{\mathbb{C}^d}(\cdot)$, *up to an exponential factor* e^{Θ}.
(*ii*) *Since we are dealing with first-order ODE systems, one needs to use the modified determinant* $\det_2(\cdot)$.

Remark 4. *There is nothing special about Schrödinger operators in Examples 1 and 2: The same applies to one-dimensional Dirac-type operators on a half-line as well as the whole real line, and to Jacobi operators* (*i.e., tri-diagonal matrices related to orthogonal polynomials on* $\mathbb{R}$) *and CMV operators* (*i.e., five-diagonal matrices related to orthogonal polynomials on the unit circle* S^1) *on a half-lattice as well as the lattice* $\mathbb{Z}$.

Semiseparable Integral Kernels, Reduction of Infinite Determinants to Finite Ones

Suppose $A\colon \mathcal{H}_1 \to \mathcal{H}_2$, $B\colon \mathcal{H}_2 \to \mathcal{H}_1$ with $AB \in \mathcal{B}_1(\mathcal{H}_2)$ and $BA \in \mathcal{B}_1(\mathcal{H}_1)$. Then,

$$\det(I_{\mathcal{H}_2} - AB) = \det(I_{\mathcal{H}_1} - BA).$$

Note: $\mathcal{H}_1$ and $\mathcal{H}_2$ may have different dimensions (finite and infinite).

Example: $m \times m$ matrix-valued, semi-separable integral kernels on $(a,b) \subseteq \mathbb{R}$

$$K(x,x') = \begin{cases} f_1(x)g_1(x'), & a < x' < x < b, \\ f_2(x)g_2(x'), & a < x < x' < b, \end{cases}$$

associated with the Hilbert–Schmidt operator K in $L^2((a,b);dx)^m$,

$$(Kf)(x) = \int_a^b dx'\, K(x,x')f(x'), \quad f \in L^2((a,b);dx)^m,\ m \in \mathbb{N},$$

assuming

$$f_j \in L^2((a,b);dx)^{m\times n_j},\ g_j \in L^2((a,b);dx)^{n_j\times m}, \quad n_j \in \mathbb{N},\ j=1,2. \tag{3}$$

Note: Green's matrices and resolvent operators associated with closed (matrix-valued) ordinary differential operators on arbitrary intervals (finite or infinite) on the real line are always of this semi-separable form.

Decompose K into a Volterra operator H_a and a finite-rank operator QR:

$$K = H_a + QR,$$

where

$$(H_a f)(x) = \int_a^x dx'\, H(x,x') f(x'), \quad f \in L^2((a,b);dx)^m,$$

$$H(x,x') = f_1(x)g_1(x') - f_2(x)g_2(x'), \quad a < x' < x < b$$

and

$$Q\colon \mathbb{C}^{n_2} \to L^2((a,b);dx)^m, \quad (Q\underline{u})(x) = f_2(x)\underline{u}, \quad \underline{u} \in \mathbb{C}^{n_2},$$

$$R\colon L^2((a,b);dx)^m \to \mathbb{C}^{n_2}, \; (Rf) = \int_a^b dx'\, g_2(x')f(x'), \; f \in L^2((a,b);dx)^m$$

Observing

$$I - \alpha K = (I - \alpha H_a)[I - \alpha(I - \alpha H_a)^{-1}QR], \quad \alpha \in \mathbb{C},$$

assuming that K is a trace class operator, $K \in \mathcal{B}_1(L^2((a,b);dx)^m)$, one computes,

$$\begin{aligned} \det(I - \alpha K) &= \det(I - \alpha H_a)\det(I - \alpha(I - \alpha H_a)^{-1}QR) \\ &= \det(I - \alpha(I - \alpha H_a)^{-1}QR) \\ &= \det{}_{\mathbb{C}^{n_2}}(I_{n_2} - \alpha R(I - \alpha H_a)^{-1}Q). \end{aligned}$$

Note: The Fredholm determinant of $I - \alpha K$ is reduced to a finite-dimensional determinant induced by the finite rank operator QR.

Now introduce the $n \times n$ matrix A $(n = n_1 + n_2)$

$$A(x) = \begin{pmatrix} g_1(x)f_1(x) & g_1(x)f_2(x) \\ -g_2(x)f_1(x) & -g_2(x)f_2(x) \end{pmatrix},$$

and the Volterra integral equations

$$\hat{f}_1(x,\alpha) = f_1(x) - \alpha \int_x^b dx'\, H(x,x')\hat{f}_1(x',\alpha),$$
$$\hat{f}_2(x,\alpha) = f_2(x) + \alpha \int_a^x dx'\, H(x,x')\hat{f}_2(x',\alpha), \quad \alpha \in \mathbb{C}$$

with solutions $\hat{f}_j(\cdot,\alpha) \in L^2((a,b);dx)^{m\times n_j}$, $j = 1,2$. Then, the first-order $n \times n$ system of differential equations

$$U'(x,\alpha) = \alpha A(x) U(x,\alpha) \text{ for a.e. } x \in (a,b) \text{ and } \alpha \in \mathbb{C}$$

permits the explicit particular solution

$$U(x,\alpha)=\begin{pmatrix} I_{n_1}-\alpha\int_x^b dx'\, g_1(x')\hat{f}_1(x',\alpha) & \alpha\int_a^x dx'\, g_1(x')\hat{f}_2(x',\alpha) \\ \alpha\int_x^b dx'\, g_2(x')\hat{f}_1(x',\alpha) & I_{n_2}-\alpha\int_a^x dx'\, g_2(x')\hat{f}_2(x',\alpha) \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det(I-\alpha K) &= \det_{\mathbb{C}^{n_2}}(I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q) \\ &= \det_{\mathbb{C}^{n_2}}\left(I_{n_2}-\alpha\int_a^b dx\, g_2(x)\hat{f}_2(x,\alpha)\right) \\ &= \det_{\mathbb{C}^{n}}(U(b,\alpha)), \end{aligned}$$

a relatively new result in this generality.

Analogous results hold for 2-modified Fredholm determinants in the case where K is only assumed to be a Hilbert–Schmidt operator.

Comment: Again this is an interesting reduction and an infinite determinant, $\det(\cdot)$, is reduced to an $n_2 \times n_2$ determinant, $\det_{\mathbb{C}^{n_2}}(\cdot)$, etc.

An Application of Perturbation Determinants to Schrödinger Operators in Dimensions $n = 1, 2, 3$

Hypothesis 5. *Suppose* $n = 1$ *and* $V \in L^1((0,\infty); dx)$.

Let $H_{0,+}^D$ and $H_{0,+}^N$ denote the Dirichlet and Neumann Laplacians, $-d^2/dx^2$, and let V be the multiplication operator by the element $V \in L^1((0,\infty); dx)$.

Introduce $H_+^D = H_{0,+}^D + V$, $H_+^N = H_{0,+}^N + V$, and the fundamental system of solutions $\phi_+(z,\cdot)$, $\theta_+(z,\cdot)$ and the Jost solution $f_+(z,\cdot)$ of $-\psi''(z,x) + V(x)\psi(z,x) = z\psi(z,x)$, $z \in \mathbb{C}\backslash\{0\}$, $x \geq 0$, by

$$\phi_+(z,x) = \frac{\sin(z^{1/2}x)}{z^{1/2}} + \int_0^x dx' \, \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} \, V(x')\phi_+(z,x'),$$

$$\theta_+(z,x) = \cos(z^{1/2}x) + \int_0^x dx' \, \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} \, V(x')\theta_+(z,x'),$$

$$f_+(z,x) = e^{iz^{1/2}x} - \int_x^\infty dx' \, \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} \, V(x')f_+(z,x').$$

Moreover, let $u = \exp(i\arg(V))\,|V|^{1/2}$ and $v = |V|^{1/2}$ s.t. $V = uv$. Denote by I_+ the identity operator in $L^2((0,\infty);dx)$.

Lemma 6 (R. Jost and A. Pais, 1951)**.** *Assume Hypothesis 5 and* $z \in \mathbb{C}\backslash[0,\infty)$ *with* $\operatorname{Im}(z^{1/2}) > 0$*. Then*

$$\det\Big(I_+ + \overline{u(H_{0,+}^D - zI_+)^{-1}v}\Big) = W(f_+(z,\cdot),\phi_+(z,\cdot)) = f_+(z,0),$$

$$\det\Big(I_+ + \overline{u(H_{0,+}^N - zI_+)^{-1}v}\Big) = \frac{W(f_+(z,\cdot),\theta_+(z,\cdot))}{-iz^{1/2}} = \frac{f_+'(z,0)}{iz^{1/2}},$$

where $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ *denotes the* **Wronskian** *of* f *and* g*.*

Comment: An infinite determinant, $det(\cdot)$, is reduced to a simple Wronski determinant, $W(\cdot,\cdot)$.

A straightforward generalization of this lemma to higher dimensions appears to be difficult, but the following formula for the ratio of determinants permits a direct extension to dimensions $n = 2, 3$.

Theorem 7. *Assume Hypothesis 5 and let* $z \in \mathbb{C} \backslash \sigma(H_+^D)$ *with* $\operatorname{Im}(z^{1/2}) > 0$. *Then,*

$$\frac{\det\Big(I_+ + \overline{u(H_{0,+}^N - zI_+)^{-1}v}\Big)}{\det\Big(I_+ + \overline{u(H_{0,+}^D - zI_+)^{-1}v}\Big)} = \frac{f_+'(z,0)}{iz^{1/2} f_+(z,0)} = \frac{m_+^D(z)}{m_{0,+}^D(z)}$$

$$= 1 - \Big(\overline{\gamma_N (H_+^D - zI_+)^{-1} V \big[\gamma_D (H_{0,+}^N - \overline{z} I_+)^{-1}\big]^*}\Big) 1,$$

where γ_D *and* γ_N *represent the boundary* **Dirichlet** *and* **Neumann trace operators** *(i.e., functionals in* $d = 1$*),*

$$\gamma_D : \begin{cases} C([0,\infty)) \to \mathbb{C}, \\ \qquad g \mapsto g(0), \end{cases} \qquad \gamma_N : \begin{cases} C^1([0,\infty)) \to \mathbb{C}, \\ \qquad h \mapsto -h'(0). \end{cases}$$

Here $m_+^D(z)$ *is the* **Weyl–Titchmarsh function** *(i.e., the* **Dirichlet to Neumann map***) of* H_+^D *and* $m_{0,+}^D(z) = iz^{1/2}$ *is that of* $H_{0,+}^D$.

Hypothesis 8. *Let* $n = 2, 3$.

(i) *Assume that* $\Omega \subset \mathbb{R}^n$ *is an open nonempty set with a compact, nonempty boundary* $\partial\Omega$ *such that one of the following conditions holds:*

(α) Ω *is of class* $C^{1,r}$ *for some* $1/2 < r < 1$.

(β) Ω *is convex.*

(γ) Ω *is a Lipschitz domain satisfying a uniform exterior ball condition* (*UEBC*).

(ii) *Suppose that* $V \in L^p(\Omega; d^n x)$, *where* $4/3 < p \leq 2$ *in the case* $n = 2$ *and* $3/2 < p \leq 2$ *in the case* $n = 3$.

Note: $\partial\Omega$ is assumed to be compact, but Ω may be unbounded in connection with conditions (α) or (γ).

In the following, $d\sigma^{n-1}$ denotes the surface measure on $\partial\Omega$.

Let γ_D denote the **Dirichlet trace operator**,

$$\gamma_D \colon H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\sigma), \quad 1/2 < s < 3/2,$$

$$\gamma_D u = u|_{\partial\Omega}, \quad u \in C(\overline{\Omega}),$$

and γ_N denote the **Neumann trace operator**,

$$\gamma_N = \nu \cdot \gamma_D \nabla \colon H^{s+1}(\Omega) \to L^2(\partial\Omega; d^{n-1}\sigma), \quad 1/2 < s < 3/2,$$

where ν denotes the outward pointing unit normal vector to $\partial\Omega$.

Actually, we also need to go further and introduce a (weak) Neumann trace operator $\widetilde{\gamma}_N$ by

$$\widetilde{\gamma}_N \colon \{u \in H^1(\Omega) \,|\, \Delta u \in (H^1(\Omega))^*\} \to H^{-1/2}(\partial\Omega), \quad \widetilde{\gamma}_N u = \nu \cdot \nabla u.$$

(Some details are omitted.....)

$\widetilde{\gamma}_N$ is an extension of γ_N.

Given Hypothesis 8 (i), we introduce the self-adjoint and nonnegative Dirichlet and Neumann Laplacians $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ in $L^2(\Omega; d^n x)$, associated with the domain Ω by,

$$H^D_{0,\Omega} = -\Delta, \quad \mathrm{dom}(H^D_{0,\Omega}) = \{u \in H^2(\Omega) \,|\, \gamma_D u = 0\},$$
$$H^N_{0,\Omega} = -\Delta, \quad \mathrm{dom}(H^N_{0,\Omega}) = \{u \in H^2(\Omega) \,|\, \gamma_N u = 0\}.$$

Let $u = \exp(i \arg(V)) \, |V|^{1/2}$, $v = |V|^{1/2}$ such that $V = uv$.

We denote by I_Ω and $I_{\partial\Omega}$ the identity operators in $L^2(\Omega; d^n x)$ and $L^2(\partial\Omega; d\sigma)$.

Applying abstract perturbation theory based on factorizations of V (Kato 1966), one can introduce the operators H^D_Ω and H^N_Ω which are closed, densely defined distinguished extensions of $H^D_{0,\Omega} + V$ on $\mathrm{dom}(H^D_{0,\Omega}) \cap \mathrm{dom}(V)$ and $H^N_{0,\Omega} + V$ on $\mathrm{dom}(H^N_{0,\Omega}) \cap \mathrm{dom}(V)$:

$$
\begin{aligned}
(H_{\Omega}^{D,N} - zI_{\Omega})^{-1} &= (H_{0,\Omega}^{D,N} - zI_{\Omega})^{-1} \\
&\quad - \overline{(H_{0,\Omega}^{D,N} - zI_{\Omega})^{-1} v} \Big[I_{\Omega} + \overline{u (H_{0,\Omega}^{D,N} - zI_{\Omega})^{-1} v} \Big]^{-1} u (H_{\Omega}^{D,N} - zI_{\Omega})^{-1}.
\end{aligned}
$$

Typical results required in the following:

Using potential theoretic notions like single and double layer potentials, Hypothesis 8 (i) implies the inclusions:

$$
\operatorname{dom}\big(H_{0,\Omega}^{D}\big) \subset H^{2}(\Omega), \quad \operatorname{dom}\big(H_{0,\Omega}^{N}\big) \subset H^{2}(\Omega).
$$

These inclusions and methods based on real interpolation spaces then allow one to prove:

Lemma 9. *Assume Hypothesis 8 (i). Then the following mapping properties hold for all $q \in [0,1]$ and $z \in \mathbb{C} \backslash [0,\infty)$,*

$$
\big(H_{0,\Omega}^{D} - zI_{\Omega}\big)^{-q}, \ \big(H_{0,\Omega}^{N} - zI_{\Omega}\big)^{-q} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^{2q}(\Omega)\big). \tag{4}
$$

Lemma 10. *Assume Hypothesis 8 (i) and let* $\varepsilon \in (0,1]$*,* $z \in \mathbb{C}\backslash[0,\infty)$*. Then,*

$$\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-\frac{3+\varepsilon}{4}}, \gamma_D\big(H_{0,\Omega}^N - zI_\Omega\big)^{-\frac{1+\varepsilon}{4}} \in \mathcal{B}\big(L^2(\Omega), L^2(\partial\Omega)\big).$$

Lemma 11. *Suppose* Ω *is an open nonempty Lipschitz domain with a compact nonempty boundary* $\partial\Omega$*. Then, the following mapping properties hold,*

$$\big(\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big)^* : H^1(\partial\Omega) \to H^{3/2}(\Omega), \quad z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}^D),$$

$$\big(\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big)^* : L^2(\partial\Omega; d^{n-1}\sigma) \to H^{3/2}(\Omega), \quad z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}^N).$$

Moreover, we need:

Lemma 12. *Assume Hypothesis 8 (i) and let* $2 \leq p$, $(n/2p) < q \leq 1$, $f \in L^p(\Omega; d^n x)$, *and* $z \in \mathbb{C}\backslash[0,\infty)$. *Then,*

$$f\big(H_{0,\Omega}^D - zI_\Omega\big)^{-q},\ f\big(H_{0,\Omega}^N - zI_\Omega\big)^{-q} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big),$$

and for some $c > 0$ *(independent of* z *and* f *)*

$$\Big\| f\big(H_{0,\Omega}^D - zI_\Omega\big)^{-q} \Big\|_{\mathcal{B}_p(L^2(\Omega;d^n x))}^2 \leq c\left(1 + \frac{|z|^{2q}+1}{\operatorname{dist}\big(z, \sigma\big(H_{0,\Omega}^D\big)\big)^{2q}}\right) \big\|\big(|\cdot|^2 - z\big)^{-q}\big\|_{L^p(\mathbb{R}^n;d^n x)}^2 \|f\|_{L^p(\Omega;d^n x)}^2,$$

$$\Big\| f\big(H_{0,\Omega}^N - zI_\Omega\big)^{-q} \Big\|_{\mathcal{B}_p(L^2(\Omega;d^n x))}^2 \leq c\left(1 + \frac{|z|^{2q}+1}{\operatorname{dist}\big(z, \sigma\big(H_{0,\Omega}^N\big)\big)^{2q}}\right) \big\|\big(|\cdot|^2 - z\big)^{-q}\big\|_{L^p(\mathbb{R}^n;d^n x)}^2 \|f\|_{L^p(\Omega;d^n x)}^2.$$

A **first principal result**, the analog of the second half of Theorem 7 for dimensions 2 and 3:

Theorem 13. *Assume Hypothesis 8 and let* $z \in \mathbb{C} \backslash \big(\sigma(H_{\Omega}^{D}) \cup \sigma(H_{0,\Omega}^{D}) \cup \sigma(H_{0,\Omega}^{N})\big)$. *Then,*

$$\frac{\det{}_2\Big(I_{\Omega} + \overline{u(H_{0,\Omega}^{N} - zI_{\Omega})^{-1}v}\Big)}{\det{}_2\Big(I_{\Omega} + \overline{u(H_{0,\Omega}^{D} - zI_{\Omega})^{-1}v}\Big)} \tag{5}$$

$$= \det{}_2\Big(I_{\partial\Omega} - \overline{\gamma_N (H_{\Omega}^{D} - zI_{\Omega})^{-1} V \big[\gamma_D (H_{0,\Omega}^{N} - \overline{z} I_{\Omega})^{-1}\big]^{*}}\Big) e^{\operatorname{tr}(T(z))},$$

where

$$T(z) = \overline{\gamma_N (H_{0,\Omega}^{D} - zI_{\Omega})^{-1} V (H_{\Omega}^{D} - zI_{\Omega})^{-1} V \big[\gamma_D (H_{0,\Omega}^{N} - \overline{z} I_{\Omega})^{-1}\big]^{*}}.$$

Comment: The ratio of "$\det_2(\cdot)$" in $L^2(\Omega)$ is reduced to "$\det_2(\cdot)$" in $L^2(\partial\Omega)$. The presence of the exponential factor follows from

$$\det{}_2(I - A) = \det(I - A) e^{\operatorname{tr}(A)}, \quad A \in \mathcal{B}_1(\mathcal{H}).$$

Basic ingredients in the proof of Theorem 12:

Introduce the Birman–Schwinger kernel,

$$K_D(z) = -\overline{u(H_{0,\Omega}^D - zI_\Omega)^{-1}v}, \quad K_N(z) = -\overline{u(H_{0,\Omega}^N - zI_\Omega)^{-1}v}$$

and rewrite the left-hand side of (5) as

$$\frac{\det{}_2\big(I_\Omega + \overline{u(H_{0,\Omega}^N - zI_\Omega)^{-1}v}\big)}{\det{}_2\big(I_\Omega + \overline{u(H_{0,\Omega}^D - zI_\Omega)^{-1}v}\big)} = \frac{\det{}_2\big(I_\Omega - K_N(z)\big)}{\det{}_2\big(I_\Omega - K_D(z)\big)}$$

$$= \det{}_2\big(I_\Omega + (K_D(z) - K_N(z))[I_\Omega - K_D(z)]^{-1}\big)$$

$$\times \exp\big(\operatorname{tr}\big((K_D(z) - K_N(z))K_D(z)[I_\Omega - K_D(z)]^{-1}\big)\big).$$

Lemma 14 (A Krein-type resolvent formula).

$$(H_{0,\Omega}^D - zI_\Omega)^{-1} - (H_{0,\Omega}^N - zI_\Omega)^{-1}$$

$$= \big[\gamma_D(H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\big]^*\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-1}.$$

Thus,

$$K_D(z) - K_N(z) = -\overline{u\big[(H_{0,\Omega}^D - zI_\Omega)^{-1} - (H_{0,\Omega}^N - zI_\Omega)^{-1}\big]v}$$
$$= -\left[\overline{\gamma_D(H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\overline{u}}\right]^* \overline{\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-1}v}.$$

Strategy:

- temporarily assume $V \in L^p(\Omega; d^n x) \cap L^\infty(\Omega; d^n x)$,
- appropriately interchange factors under the determinant, arrive at the final determinant formula in the theorem,
- use an approximation argument to remove the temporary assumption $V \in L^\infty(\Omega; d^n x)$.

Corollary 15 (A Reduction Principle). *The following observation yields a simple application of formula* (5). *By the general Birman–Schwinger principle,*

$$\textit{for all } z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}^N), \ \textit{one has } z \in \sigma(H_\Omega^N)$$

if and only if

$$\det{}_2\Big(I_\Omega + \overline{u(H_{0,\Omega}^N - zI_\Omega)^{-1}v}\Big) = 0.$$

Thus, it follows from (5) *that*

$$\textit{for all } z \in \mathbb{C}\backslash\big(\sigma(H_\Omega^D) \cup \sigma(H_{0,\Omega}^D) \cup \sigma(H_{0,\Omega}^N)\big), \ \textit{one has } z \in \sigma(H_\Omega^N)$$

if and only if

$$\det{}_2\Big(I_{\partial\Omega} - \overline{\gamma_N(H_\Omega^D - zI_\Omega)^{-1}V\big[\gamma_D(H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\big]^*}\Big) = 0.$$

Comment: The eigenvalue problem in $L^2(\Omega)$ has been reduced to one in $L^2(\partial\Omega)$.

The Connection to Dirichlet-to-Neumann Maps

Lemma 16. *Assume Hypothesis 8 (i) and* $V \in L^2(\Omega) \cap L^p(\Omega)$, $p > 2$. *Then there exist the* **Neumann-to-Dirichlet maps** $M_{0,\Omega}^N(z)$ *and* $M_\Omega^N(z)$ *associated with* $(-\Delta - z)$ *and* $(-\Delta - z + V)$ *on* Ω *defined as*

$$M_{0,\Omega}^N(z) \colon \begin{cases} L^2(\partial\Omega; d\sigma^{n-1}) \to H^1(\partial\Omega), \\ \qquad\qquad f \mapsto \gamma_D u_0^N, \end{cases} \qquad z \in \mathbb{C} \backslash \sigma(H_{0,\Omega}^N),$$

where $u_0^N \in H^{3/2}(\Omega)$ *is the unique solution of*

$$(-\Delta - z) u_0^N = 0 \;\; on \;\; \Omega, \quad \widetilde{\gamma}_N u_0^N = f \;\; on \;\; \partial\Omega,$$

and

$$M_\Omega^N(z) \colon \begin{cases} L^2(\partial\Omega; d\sigma^{n-1}) \to H^1(\partial\Omega), \\ \qquad\qquad f \mapsto \gamma_D u^N, \end{cases} \qquad z \in \mathbb{C} \backslash \sigma(H_\Omega^N),$$

where $u^N \in H^{3/2}(\Omega)$ *is the unique solution of*

$$(-\Delta - z + V) u^N = 0 \;\; on \;\; \Omega, \quad \widetilde{\gamma}_N u^N = f \;\; on \;\; \partial\Omega.$$

The **Dirichlet-to-Neumann maps**

$$M_{0,\Omega}^{D}(z) = -M_{0,\Omega}^{N}(z)^{-1} \text{ and } M_{\Omega}^{D}(z) = -M_{\Omega}^{N}(z)^{-1}$$

have the following mapping properties

$$M_{0,\Omega}^{D}(z) : H^1(\partial\Omega) \to L^2(\partial\Omega; d\sigma^{n-1}), \quad z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}^{D}),$$
$$M_{\Omega}^{D}(z) : H^1(\partial\Omega) \to L^2(\partial\Omega; d\sigma^{n-1}), \quad z \in \mathbb{C}\backslash\sigma(H_{\Omega}^{D}).$$

Note: $M_{0,\Omega}^{N}(z), M_{\Omega}^{N}(z) \in \mathcal{B}_\infty(L^2(\partial\Omega; d\sigma^{n-1}))$ are compact, but $M_{0,\Omega}^{D}(z), M_{\Omega}^{D}(z)$ are densely defined and unbounded operators in $L^2(\partial\Omega; d\sigma^{n-1})$ with domain $H^1(\partial\Omega)$.

$M_{\Omega}^{D}(z)$ and $M_{\Omega}^{N}(z)$ were discussed by Amrein and Pearson (2004) for Ω = the interior or exterior of a ball, and $V \in L^\infty(\mathbb{R}^3)$.

Our methods are quite different since we consider rough Ω (Lipschitz + a bit more) and $V \in L^p(\mathbb{R}^n)$, $n = 2, 3$, for nearly optimal values of p.

In addition,

$$M_{0,\Omega}^{N}(z) = \gamma_D\big[\gamma_D\big((H_{0,\Omega}^{N} - \overline{z}I_\Omega)^{-1}\big)\big]^*,$$

$$M_{\Omega}^{N}(z) = \gamma_D\big[\gamma_D\big((H_{\Omega}^{N} - zI_\Omega)^{-1}\big)^*\big]^*,$$

$$M_{0,\Omega}^{D}(z) = \widetilde{\gamma}_N\big[\gamma_N\big((H_{0,\Omega}^{D} - \overline{z}I_\Omega)^{-1}\big)\big]^*,$$

$$M_{\Omega}^{D}(z) = \widetilde{\gamma}_N\big[\gamma_N\big((H_{\Omega}^{D} - zI_\Omega)^{-1}\big)^*\big]^*.$$

These formulas permit the extension of $M_\Omega^D(z)$ and $M_\Omega^N(z)$ to the case $V \in L^p(\Omega; d^n x)$, where $4/3 < p \leq 2$ if $n = 2$ and $3/2 < p \leq 2$ if $n = 3$, as in Hypothesis 8 (ii).

A Key Identity:

$$M_\Omega^D(z) M_{0,\Omega}^D(z)^{-1} - I_{\partial\Omega}$$
$$= -\overline{\gamma_N (H_\Omega^D - zI_\Omega)^{-1} V \big[\gamma_D (H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\big]^*} \in \mathcal{B}_2(L^2(\partial\Omega; d\sigma^{n-1})).$$

Thus, the determinant formula in Theorem 13 can finally be written as:

Theorem 17. *Assume Hypothesis 8 and let* $z \in \mathbb{C}\backslash\big(\sigma(H_\Omega^D) \cup \sigma(H_{0,\Omega}^D) \cup \sigma(H_{0,\Omega}^N)\big)$. *Then,*

$$\frac{\det{}_2\Big(I_\Omega + \overline{u(H_{0,\Omega}^N - zI_\Omega)^{-1}v}\Big)}{\det{}_2\Big(I_\Omega + \overline{u(H_{0,\Omega}^D - zI_\Omega)^{-1}v}\Big)}$$

$$= \det{}_2\Big(I_{\partial\Omega} - \overline{\gamma_N(H_\Omega^D - zI_\Omega)^{-1}V\big[\gamma_D(H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\big]^*}\Big)e^{\operatorname{tr}(T(z))}$$

$$= \det{}_2\Big(M_\Omega^D(z)M_{0,\Omega}^D(z)^{-1}\Big)e^{\operatorname{tr}(T(z))},$$

where

$$T(z) = \overline{\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-1}V(H_\Omega^D - zI_\Omega)^{-1}V\big[\gamma_D(H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\big]^*}.$$