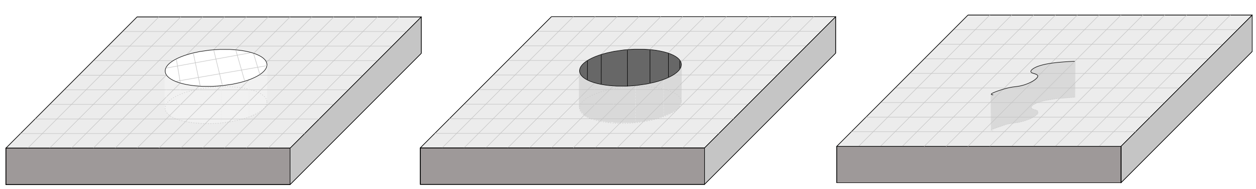
Trapped modes in elastic media

Clemens Förster
Institut für Analysis, Dynamik und Modellierung
Universität Stuttgart

June 2006



Elasticity operator

Consider the elasticity operator for zero Poisson ratio

$$A_0 = -(\Delta + \operatorname{grad}\operatorname{div}) \quad \text{in} \quad L^2(\mathbb{R}^2 \times J; \mathbb{C}^3),$$

$J := (-\frac{\pi}{2}, \frac{\pi}{2})$, with stress-free boundary conditions.

The corresponding quadratic form is

$$a_0[u] = 2\int_{\mathbb{R}^2 \times J} |\epsilon(u)|^2 \, \mathrm{d}x, \quad u \in H^1(\mathbb{R}^2 \times J; \mathbb{C}^3)$$

where $\epsilon(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$.

For $f \in L^\infty(\mathbb{R}^2; [0,1])$ compactly supported and $\alpha \in (0,1)$ consider the perturbed operator A_α corresponding to

$$a_\alpha[u] = 2\int_{\mathbb{R}^2 \times J} (1 - \alpha f)|\epsilon(u)|^2 \, \mathrm{d}x, \quad u \in H^1(G; \mathbb{C}^3).$$

We have

- $\sigma(A_0) = [0, \infty)$, purely absolutely continuous.
- $\sigma(A_\alpha) = [0, \infty)$ with embedded eigenvalues for $\alpha \neq 0$.

We investigate these eigenvalues.

Symmetries and spectral structure

$$H_1 := \{u \in L^2(\mathbb{R}^2 \times J; \mathbb{C}^3) \mid u_{1/2} \text{ symmetric in } x_3, \\ u_3 \text{ antisymmetric in } x_3\}$$

$$H_4 := \{u \in H_1 \mid \int_J u_{1/2} \, \mathrm{d}x_3 \equiv 0 \text{ in } L^2(\mathbb{R}^2; \mathbb{C})\},$$

$H_4, H_4^\perp$ reduce A_α. Consider $A_\alpha^{(4)} := A_\alpha|_{D(A_\alpha)\cap H_4}$. Now

- $\sigma_e(A_\alpha^{(4)}) = [\Lambda, \infty)$ for $\alpha \in [0, 1)$, $\Lambda > 0$.
- $\sigma_d(A_\alpha^{(4)}) \subset (0, \Lambda)$ for $\alpha \in (0, 1)$.

For a closer look on $\sigma_e(A_0^{(4)})$ use Fourier transform Φ in (x_1, x_2) and consider

$$\Phi A_0^{(4)} \Phi^* = \int_{\mathbb{R}^2}^{\oplus} A^{(4)}(\xi) \, \mathrm{d}\xi.$$

Eigenvalues of $A^{(4)}(\xi)$ depend only on $|\xi|$. For $\xi := (0, r)^T$

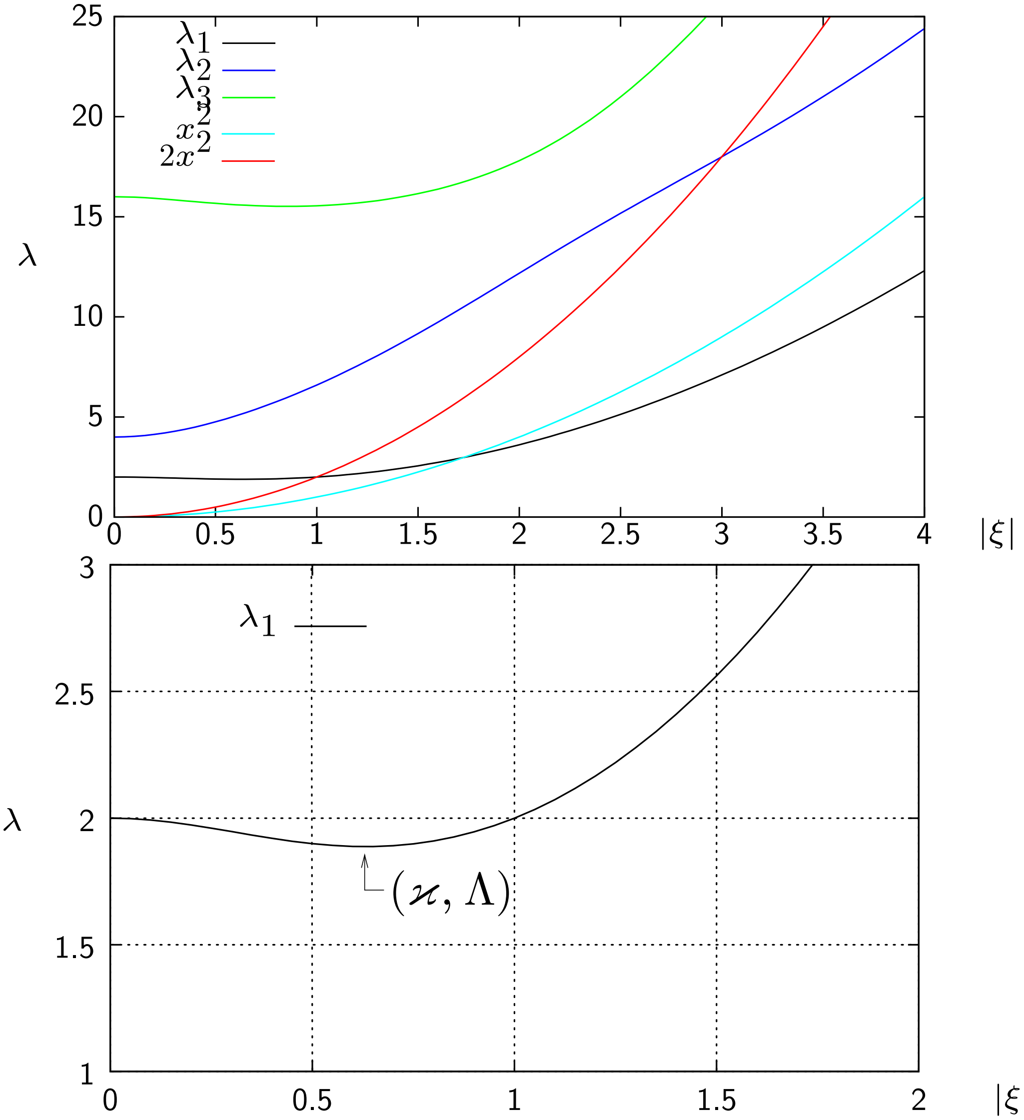
$$A^{(4)}(r) = \begin{pmatrix} -\partial_3^2 + r^2 & 0 & 0 \\ 0 & -\partial_3^2 + 2r^2 & -ir\partial_3 \\ 0 & -ir\partial_3 & -2\partial_3^2 + r^2 \end{pmatrix}$$

Eigenvalues for $-\partial_3 + r^2$ with symmetry/boundary conditions:

$$\lambda_k(r) = r^2 + 4k^2, \quad k \in \mathbb{N}.$$

Spectral structure

Eigenvalues for the remaining part: nontrivial!



$$\lambda_1(\varkappa+\varepsilon) = \Lambda + q^2\varepsilon^2 + O(\varepsilon^3) \quad \text{for} \quad \varepsilon \longrightarrow 0$$

with $\varkappa \approx 0.64, \quad \Lambda \approx 1.88, \quad q \approx 0.84.$

(Pseudo-)Eigenfunctions for spectral minimum

For all $\xi \in \mathbb{R}^2$ with $|\xi| = \varkappa$ it holds

$$-(\Delta + \operatorname{grad}\operatorname{div}) w_\xi(x) = \Lambda w_\xi(x) \quad \text{for} \quad x \in \mathbb{R}^2 \times J$$

where

$$w_\xi(x) = \begin{pmatrix} i\xi d_1(x_3) \\ d_2(x_3) \end{pmatrix} \mathrm{e}^{i\xi \cdot \binom{x_1}{x_2}}$$

and

$$d_1(x_3) := \varkappa\beta \cos(\tfrac{\pi}{2}\beta) \cos(\gamma x_3) + \frac{\gamma^2 \beta}{\varkappa} \cos(\tfrac{\pi}{2}\gamma) \cos(\beta x_3)$$

$$d_2(x_3) := -\varkappa\beta\gamma \cos(\tfrac{\pi}{2}\beta) \sin(\gamma x_3) + \varkappa\gamma^2 \cos(\tfrac{\pi}{2}\gamma) \sin(\beta x_3)$$

$$\beta := \sqrt{\Lambda - \varkappa^2}, \quad \gamma := \sqrt{\Lambda/2 - \varkappa^2}.$$

Existence of eigenvalues

Theorem 1.
If $f \neq 0$ in L^2-sense then $A_\alpha^{(4)}$ has infinitely many eigenvalues in $(0, \Lambda)$ for all $\alpha \in (0,1)$.

Proof. • Take m different w_{ξ^k}, $k = 1, \ldots, m$.

- Define $\zeta_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^2)$ such that $\zeta_\varepsilon = 1$ in $\operatorname{supp} f$ and

$$\int_{\mathbb{R}^2} |\nabla \zeta_\varepsilon|^2 \, \mathrm{d}x \to 0, \quad \varepsilon \to 0.$$

- Define $u_k^{(\varepsilon)}(x) := \zeta_\varepsilon(x) w_{\xi^k}(x)$ for $x \in \mathbb{R}^2 \times J$,

$$u_\eta := \textstyle\sum_{k=1}^m \eta_k u_k^{(\varepsilon)}, \ \eta \in \mathbb{C}^m$$

$$\mathcal{E}_m^{(\varepsilon)} := \operatorname{span}\{u_k^{(\varepsilon)} \mid k = 1, \ldots, m\}.$$

- $\dim \mathcal{E}_m^{(\varepsilon)} = m$.
- It holds independently of sufficiently small ε

$$\inf_{|\eta|=1} 2 \int_{\mathbb{R}^2 \times J} f |\epsilon(u_\eta)|^2 \, \mathrm{d}x \geq c_0 > 0.$$

- For sufficiently small $\varepsilon > 0$ and $|\eta| = 1$ it holds

$$a_0^{(4)}[u_\eta] - \Lambda \|u_\eta\|^2 \leq C \int_{\mathbb{R}^2} |\nabla \zeta_\varepsilon|^2 \, \mathrm{d}(x_1, x_2) < \alpha c_0$$

- We obtain

$$
\begin{aligned}
a_\alpha^{(4)}[u_\eta] = a_0^{(4)}[u_\eta] - \Lambda\|u_\eta\|^2 - 2\alpha \int_{\mathbb{R}^2\times J} f|\epsilon(u_\eta)|^2 \,\mathrm{d}x \\
+ \Lambda\|u_\eta\|^2 \\
< \alpha c_0|\eta|^2 - \alpha c_0|\eta|^2 + \Lambda\|u_\eta\|^2
\end{aligned}
$$

for ε small enough.

□

Estimates for the eigenvalues

Denote by $\varkappa_k(\alpha)$ the eigenvalues of $A_\alpha^{(4)}$ below Λ in non-decreasing order. Let $\lambda_k(K)$ be the eigenvalues of a certain compact integral operator K in non-increasing order.

Theorem 2.

$$\ln(\Lambda - \varkappa_k(\alpha)) = -2k \ln k + o(k \ln k) \quad as \quad k \to \infty.$$

Theorem 3.

$$\varkappa_k(\alpha) = \Lambda - \alpha^2 (\Lambda \pi \lambda_k(K))^2 + o(\alpha^2) \quad as \quad \alpha \to 0.$$

Strategy of the proofs:

- Apply Birman-Schwinger principle.
- Reduce Birman-Schwinger operator to spectral minimum.
- Estimate eigenvalues of reduced operator.
- Estimate remainder terms.

Proofs are based on:

- Laptev, Safronov, Weidl: Bound state asymptotics for elliptic operators with strongly degenerating symbols. (2002)
- Förster, Weidl: Trapped modes for an elastic strip with perturbation of the material properties. (2006)

Birman-Schwinger principle

Consider

$$Qu := \tfrac{1}{\sqrt{2}}(\nabla u + (\nabla u)^T), \quad u \in H^1(\mathbb{R}^2 \times J; \mathbb{C}^3) \cap H_4.$$

Then $A_0^{(4)} = Q^*Q$. Define partial isometry $U = Q(A_0^{(4)})^{-\frac{1}{2}}$. Then the Birman-Schwinger operator is given by

$$\mathcal{Y}_\alpha(\tau) := \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}} V_\alpha \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}} \qquad (1)$$

in H_4 for $\tau \in (0, \Lambda)$ and $\alpha \in (0, 1)$ where

$$V_\alpha = U^* \sqrt{\alpha f}(I - \alpha \sqrt{f}\, UU^* \sqrt{f})^{-1} \sqrt{\alpha f}\, U.$$

Lemma 4.
For $\tau \in (0, \Lambda)$ and $\alpha \in (0, 1)$ it holds

$$n_-(\Lambda - \tau, A_\alpha^{(4)}) = n_+(1, \mathcal{Y}_\alpha(\tau)).$$

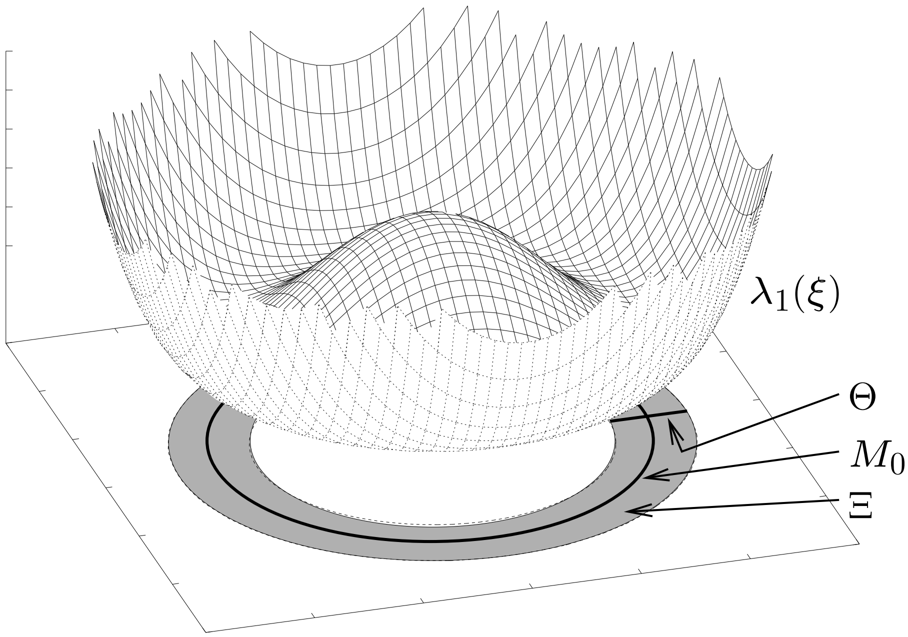
Proof. Use Glazman-lemma and transformations like in the proof of the Birman-Schwinger principle. □

Because of

$$U^* \alpha f U \le V_\alpha \le U^* \tfrac{\alpha}{1-\alpha} f U$$

we can consider $\mathcal{Y}(\tau)$, which is (1) with V_α replaced by $U^* f U$.

Reduction to spectral minimum



$$\mathcal{Y}(\tau) := \mathcal{W}\mathcal{W}^*, \quad \mathcal{W} = \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}} U^* \sqrt{f}$$

$$\mathcal{L}(\tau) := F^* \mathcal{K}\mathcal{K}^* F \oplus \mathbb{O}, \quad \mathcal{K} = \left(\frac{\Lambda - \tau}{\lambda^2 + \tau}\right)^{\frac{1}{2}} \otimes X$$

Π_c : spectral projection on $\lambda_1(\Xi)$

$F : \Pi_c L^2(\mathbb{R}^2 \times J; \mathbb{C}^3) \to L^2(\Theta) \otimes L^2(M_0, d\mu_0)$

$\mathcal{K} : H_4^\times \to L^2(\Theta) \otimes L^2(M_0, d\mu_0)$

$$Xu := \operatorname{ess} \lim_{\lambda \to 0} (F \Pi_c U^* \sqrt{f} u)(\lambda)$$

Proof of the accumulation rate

- $K = XX^*$ is integral operator on $L^2(M_0, d\mu_0)$ with kernel $k(\eta, \xi) =$

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times J} f(x) \mathrm{e}^{i(\eta-\xi)\cdot\binom{x_1}{x_2}} \langle \varphi(\xi, x_3), \varphi(\eta, x_3) \rangle_{\mathbb{C}^{3\times 3}} \, \mathrm{d}x.$$

- For $K \in \mathfrak{S}_\infty, \mathcal{Y} : \mathbb{R}_+ \to \mathfrak{S}_\infty$ and $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ define

$$\Delta_\pi(K) = \limsup_{\tau \to 0} \pi(\tau) n_+(\tau, K)$$

$$\Delta_\pi(\mathcal{Y}) = \limsup_{\tau \to 0} \pi(\tau) n_+(1, \mathcal{Y}(\tau))$$

- Direct calculation gives for certain $\pi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\Delta_\pi(K) = 1.$$

- For $\varrho(\tau) = \pi(\sqrt{\tau})$ we get

$$\Delta_\varrho(\mathcal{Y}_\alpha) = \Delta_\varrho(\mathcal{Y}) = \Delta_\varrho(\mathcal{L}) = \Delta_\pi(K) = 1.$$

- Birman-Schwinger principle yields

$$\limsup_{\tau \to 0} \varrho(\tau) n_-(\Lambda - \tau, A_\alpha^{(4)}) = \Delta_\varrho(\mathcal{Y}_\alpha) = 1.$$

$\square$

Proof for small coupling asymptotics

- $K = XX^*$ is integral operator on $L^2(M_0, d\mu_0)$ with kernel $k(\eta, \xi) =$

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times J} f(x) \mathrm{e}^{i(\eta - \xi)\cdot \binom{x_1}{x_2}} \langle \varphi(\xi, x_3), \varphi(\eta, x_3) \rangle_{\mathbb{C}^{3\times 3}} \, \mathrm{d}x.$$

- From tensor structure it follows

$$\lambda_k(\sqrt{\tau}\mathcal{L}(\tau)) = \Lambda\pi\lambda_k(K) + o(1), \quad \tau \to 0.$$

- Remainder estimate yields

$$\lambda_k(\sqrt{\tau}\mathcal{Y}(\tau)) \to \Lambda\pi\lambda_k(K), \quad \tau \to 0.$$

- Birman-Schwinger principle gives

$$\lambda_k(\alpha\mathcal{Y}(\Lambda - \varkappa_k(\alpha))) \to 1, \quad \alpha \to 0.$$

- Combining the last two equations yields

$$\alpha^{-1}\sqrt{\Lambda - \varkappa_k(\alpha)} \to \Lambda\pi\lambda_k(K), \quad \alpha \to 0.$$

□