Spectrum of a periodic operator perturbed by a small non-self adjoint operator

Denis Borisov

(jointly with Rustem Gadyl'shin)

Nuclear Physics Institute, Řež, Czech Republic

Bashkir State Pedagogical University, Ufa, Russia

1. The problem

$$\mathcal{H}_\varepsilon := \mathcal{H}_0 - \varepsilon\mathcal{L}_\varepsilon \quad \text{in} \quad L_2(\mathbb{R}); \qquad D(\mathcal{H}_\varepsilon) := W_2^2(\mathbb{R});$$

$$\mathcal{H}_0 := -\frac{d}{dx}p\frac{d}{dx} + q \quad \text{in} \quad L_2(\mathbb{R}); \qquad D(\mathcal{H}_0) := W_2^2(\mathbb{R});$$

$\varepsilon \to +0$, $p \in C^1(\mathbb{R})$, $q \in C(\mathbb{R})$, p, q are 1-periodic and real-valued, $p(x) \geqslant p_0 > 0$, $\mathcal{L}_\varepsilon : W_2^2(a_1, a_2) \to L_2(\mathbb{R}; [a_1, a_2])$, $L_2(\mathbb{R}; [a_1, a_2]) := \{u \in L_2(\mathbb{R}) : \operatorname{supp} u \subseteq [a_1, a_2]\}$,

$$\|\mathcal{L}_\varepsilon u\|_{L_2(a_1,a_2)} \leqslant C\|u\|_{W_2^2(a_1,a_2)}.$$

The operator $\mathcal{H}_0$ is *self-adjoint*, while $\mathcal{L}_\varepsilon$, $\mathcal{H}_\varepsilon$ are *not.*

Main question: *What is the behaviour of the spectrum of $\mathcal{H}_\varepsilon$ as $\varepsilon \to +0$?*

In more detail:

- *Structure and convergence of spectrum*
- *Existence of embedded eigenvalues*
- *Existence of the isolated eigenvalues and their asymptotic expansions*
- *Examples of $\mathcal{L}_\varepsilon$*

Brief history: The case

$$\mathcal{H}_\varepsilon := \mathcal{H}_0 + \varepsilon W, \quad \mathcal{H}_0 := -\frac{d^2}{dx^2} + V,$$
$$V(x+a) = V(x), \quad \int\limits_{\mathbb{R}} |W|(1+|x|)\,\mathrm{d}x < \infty,$$

was considered in *F. Gesztezy, B. Simon, A short proof of Zheludev's theorem, Trans. AMS, 1993, V. 335, No. 1, P. 329-340.* The results:

- $\sigma_{\mathrm{e}}(\mathcal{H}_\varepsilon) = \sigma_{\mathrm{e}}(\mathcal{H}_0)$ and the essential spectrum of $\mathcal{H}_\varepsilon$ has a band structure;
- each isolated eigenvalue converges to an edge of the essential spectrum as $\varepsilon \to +0$
- the discrete spectrum can accumulate at infinity only

Other obvious facts:

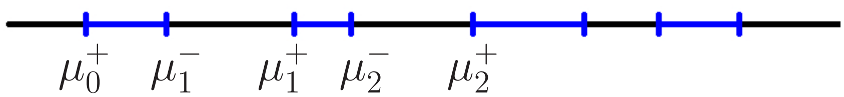
- the residual spectrum is empty
- the discrete spectrum is a countable set
- each isolated eigenvalue is simple
- there are no embedded eigenvalues

Structure and convergence of spectrum

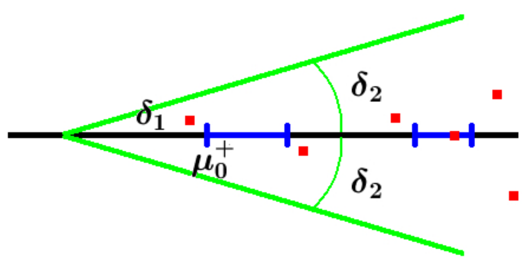
$$\mathcal{H}_\varepsilon := \mathcal{H}_0 - \varepsilon\mathcal{L}_\varepsilon, \quad \mathcal{H}_0 := -\frac{d}{dx}p\frac{d}{dx} + q,$$

$$\mathcal{L}_\varepsilon : W_2^2(a_1,a_2) \to L_2(\mathbb{R};[a_1,a_2]), \quad \|\mathcal{L}_\varepsilon u\|_{L_2(a_1,a_2)} \leqslant C\|u\|_{W_2^2(a_1,a_2)}.$$

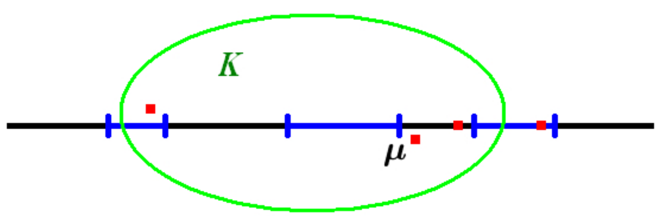
- $\mathcal{H}_\varepsilon$ is a closed operator
- $\sigma_{\mathrm{r}}(\mathcal{H}_\varepsilon) = \emptyset$, $\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon) = \sigma_{\mathrm{e}}(\mathcal{H}_0)$
- the continuous spectrum $\sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon)$ has a band structure:



- the point spectrum $\sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon)$ is a countable set and can accumulate at infinity only
- there exist functions $\delta_i(\varepsilon) \xrightarrow[\varepsilon\to+0]{} 0$, $i = 1, 2$ such that $\sigma(\mathcal{H}_\varepsilon) \subset \{\lambda : \arg(\lambda - \mu_0^+ - \delta_1(\varepsilon)) < \delta_2(\varepsilon)\}$



- $K \Subset \mathbb{C} \Rightarrow K \cap \sigma_{\mathrm{c}}(\mathcal{H}_\varepsilon) \cap \sigma_{\mathrm{p}}(\mathcal{H}_\varepsilon) = \emptyset$ if ε is small enough. $\lambda_\varepsilon \in K$ as $\varepsilon \to +0 \Rightarrow \lambda_\varepsilon \xrightarrow[\varepsilon\to+0]{} \mu$, and λ_ε is simple.



Existence of embedded eigenvalues

The operator $\mathcal{H}_\varepsilon$ ***can have*** embedded eigenvalues. Example:

$$\mathcal{L}_\varepsilon(u) := 2\alpha_\varepsilon(x) l_\varepsilon(u), \quad l_\varepsilon : W_2^2(-2\pi, 2\pi) \to \mathbb{C},$$

$$\alpha_\varepsilon(x) := c_\varepsilon \chi_{\left[-\frac{2\pi n}{k_\varepsilon}, \frac{2\pi n}{k_\varepsilon}\right]} \sin k_\varepsilon |x|, \quad n := [k_\varepsilon], \quad k_\varepsilon := \frac{\pi}{2\varepsilon^2},$$

$$c_\varepsilon := -\left(\frac{2\pi n}{k} - \frac{\sqrt{\varepsilon}}{2}\right)^{-1},$$

$$l_\varepsilon(u) := \varepsilon^{-1}\big(u'(\varepsilon^2) - u'(0)\big), \quad |l_\varepsilon(u)| \leqslant \|u''\|_{L_2(0,\sqrt{\varepsilon})},$$

$$\mathcal{H}_\varepsilon := -\frac{d^2}{dx^2} - \varepsilon \mathcal{L}_\varepsilon.$$

Then $\lambda_\varepsilon = k_\varepsilon^2 \xrightarrow[\varepsilon \to +0]{} +\infty$ is an embedded eigenvalue of $\mathcal{H}_\varepsilon$.

The operator $\mathcal{H}_\varepsilon$ ***has no*** embedded eigenvalues if

- The estimate

$$\|\mathcal{L}_\varepsilon u\|_{L_2(Q)} \leqslant C \|u\|_{W_2^2(Q)} \qquad (*)$$

is valid for each $Q \subset (a_1, a_2)$, C is independent on ε and Q.

or

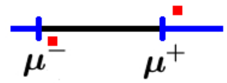
- $\mathcal{L}_\varepsilon = -A_\varepsilon(x)\dfrac{d^2}{dx^2} + \mathcal{L}_\varepsilon^{(1)}$, $\operatorname{supp} A_\varepsilon \subseteq [a_1, a_2]$, A_ε is piecewise continuously differentiable and

$$\varepsilon \max_{[a_1,a_2]} \big(|A_\varepsilon(x)| + |A'_\varepsilon(x)|\big) \xrightarrow[\varepsilon \to +0]{} 0,$$

$\mathcal{L}_\varepsilon^{(1)} : W_2^1(a_1, a_2) \to L_2(\mathbb{R}; [a_1, a_2])$, $\|\mathcal{L}_\varepsilon u\|_{L_2(a_1,a_2)} \leqslant C \|u\|_{W_2^2(a_1,a_2)}$.

If (*) holds, each eigenvalue of $\mathcal{H}_\varepsilon$ is simple.

Existence of the isolated eigenvalues and their asymptotic expansions



- Operator $\mathcal{H}_\varepsilon$ has at most one eigenvalue $\lambda_\varepsilon^\pm$ converging to $\mu^\pm$ as $\varepsilon \to +0$. If exists, this eigenvalue is simple.

Auxiliary notations:

$$\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q(x) - \lambda\right)\theta_i(x,\lambda) = 0, \quad x \in \mathbb{R}, \quad i = 1, 2,$$

$$\theta_1(0,\lambda) = 1, \quad \theta_1'(0,\lambda) = 0, \quad \theta_2(0,\lambda) = 0, \quad \theta_2'(0,\lambda) = 1,$$

$\phi_\pm = \phi_\pm(x)$ is 1-(anti)periodic solution to

$$\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q(x) - \mu^\pm\right)\phi_\pm(x) = 0, \quad x \in \mathbb{R},$$

$$\big|\phi_\pm(0)\big|^2 + \big|\phi_\pm'(0)\big|^2 = |\theta_1'(1,\mu^\pm)| + |\theta_2(1,\mu^\pm)| \neq 0.$$

$k_\pm(\varepsilon) := \pm(\mathcal{L}_\varepsilon\phi_\pm, \phi_\pm)_{L_2(a_1,a_2)}$. Assume that $|k_\pm(\varepsilon)| \geqslant C > 0$.

- If $\operatorname{Re} k_\pm(\varepsilon) > 0$, then $\lambda_\varepsilon^\pm$ exists and

$$\lambda_\varepsilon^\pm = \mu^\pm \mp \frac{\varepsilon^2 k_\pm^2(\varepsilon)}{4\gamma^\pm} + \mathcal{O}(\varepsilon^3),$$

$$\gamma^\pm := \left|\frac{d}{d\lambda}\big(\theta_1(1,\lambda) + \theta_2'(1,\lambda)\big)\big|_{\lambda=\mu^\pm}\right|.$$

- If $\operatorname{Re} k_\pm(\varepsilon) < 0$, then $\lambda_\varepsilon^\pm$ does not exist.

Examples of $\mathcal{L}_\varepsilon$

1. *Second-order differential operator:*

$$\mathcal{L}_\varepsilon = b_2(x,\varepsilon)\frac{d^2}{dx^2} + b_1(x,\varepsilon)\frac{d}{dx} + b_0(x,\varepsilon),$$
$$\operatorname{supp} b_i(\cdot,\varepsilon) \subseteq [a_1,a_2], \quad \|b_i(\cdot,\varepsilon)\|_{L_\infty(b_1,b_2)} \leqslant C,$$

Each eigenvalue of $\mathcal{H}_\varepsilon$ is simple; there are no embedded eigenvalues;

$$k_\pm(\varepsilon) = \pm\int\limits_{a_1}^{a_2} \phi_\pm \left(b_2(x,\varepsilon)\frac{d^2}{dx^2} + b_1(x,\varepsilon)\frac{d}{dx} + b_0(x,\varepsilon) \right) \phi_\pm \,\mathrm{d}x,$$

$\lambda_\varepsilon^\pm$ exists if $\operatorname{Re} k_\pm \geqslant C > 0$,

$$\lambda_\varepsilon^\pm = \mu^\pm \mp \frac{\varepsilon^2 k_\pm^2(\varepsilon)}{4\gamma^\pm} + \mathcal{O}(\varepsilon^3),$$

and $\lambda_\varepsilon^\pm$ does not exist if $\operatorname{Re} k_\pm \leqslant C < 0$.

2. *Integral operator:*

$$(\mathcal{L}_\varepsilon u)(x) = \int\limits_{a_1}^{a_2} L(x,y,\varepsilon)u(y)\,\mathrm{d}y,$$
$$\operatorname{supp} L(\cdot,\varepsilon) \subseteq [a_1,a_2]\times[a_1,a_2], \quad \|L(\cdot,\varepsilon)\|_{L_2([a_1,a_2]\times[a_1,a_2])} \leqslant C.$$

There are no embedded eigenvalues;

$$k_\pm(\varepsilon) = \pm\int\limits_{a_1}^{a_2}\int\limits_{a_1}^{a_2} \phi_\pm(x)\phi_\pm(y)L(x,y,\varepsilon)\,\mathrm{d}x\,\mathrm{d}y.$$

3. *δ-potential:* $\mathcal{H}_\varepsilon = \mathcal{H}_0 - \varepsilon\beta\delta(x-x_0)$, i.e.,

$$\mathcal{H}_\varepsilon u := \left(-\frac{d}{dx}p\frac{d}{dx}+q\right)u, \quad x \neq x_0,$$

$$u \in W_2^2(-\infty, x_0), u \in W_2^2(x_0, +\infty),$$

$$u(x_0-0) = u(x_0+0) =: u(x_0),$$

$$\frac{du}{dx}(x_0+0) - \frac{du}{dx}(x_0-0) = \frac{\beta}{p(x_0)}u(x_0)$$

The change $u(x) := v(x)\left(1+\frac{\varepsilon\beta}{2p(x_0)}|x|\chi(x)\right)$ transforms the problem to the case $\mathcal{H}_\varepsilon = \mathcal{H}_0 - \varepsilon\mathcal{L}_\varepsilon$, where $\mathcal{L}_\varepsilon$ is a second-order differential operator.

Each eigenvalue is real and simple; there are no embedded eigenvalues; if $\phi_\pm(x_0) \neq 0$, and $\pm\beta > 0$, then $\lambda_\varepsilon^\pm$ exists and

$$\lambda_\varepsilon^\pm = \mu_\pm \mp \frac{\varepsilon^2\beta^2\phi_\pm^4(x_0)}{4\gamma^\pm} + \mathcal{O}(\varepsilon^3);$$

if $\phi_\pm(x_0) \neq 0$, and $\pm\beta < 0$, then the eigenvalue $\lambda_\varepsilon^\pm$ does not exist.