On the classification of hyperbolicity and stability preservers

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Stable polynomials in n variables:

$$\mathcal{H}_n(\mathbb{C}) = \{P \in \mathbb{C}[z_1, \dots, z_n] : P(z_1, \dots, z_n) \neq 0 \text{ if } \mathfrak{Im} z_i > 0, 1 \leq i \leq n\}$$

n-SPOs = stability-preserving operators:

$$\mathcal{A}_S(n) = \{\text{linear operators } T \text{ on } \mathbb{C}[z_1, \dots, z_n] \text{ s.t. } T(\mathcal{H}_n(\mathbb{C})) \subseteq \mathcal{H}_n(\mathbb{C}) \cup \{0\}\}.$$

Real-stable polynomials in n variables:

$$\mathcal{H}_n(\mathbb{R}) = \mathcal{H}_n(\mathbb{C}) \cap \mathbb{R}[z_1, \dots, z_n].$$

n-RSPOs = real stability-preserving operators:

$$\mathcal{A}_{RS}(n) = \{\text{linear operators } T \text{ on } \mathbb{C}[z_1, \dots, z_n] \text{ s.t. } T(\mathcal{H}_n(\mathbb{R})) \subseteq \mathcal{H}_n(\mathbb{R}) \cup \{0\}\}.$$

Problem. Describe the monoids $\mathcal{A}_S(n)$ and $\mathcal{A}_{RS}(n)$, i.e., classify all n-SPOs and n-RSPOs, respectively.

$$P \in \mathcal{H}_n(\mathbb{C}) \Leftrightarrow P(\mathbf{a} + z\mathbf{v}) \in \mathcal{H}_1(\mathbb{C}) \quad \forall(\mathbf{a}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n_+.$$

$$P \in \mathcal{H}_n(\mathbb{R}) \Leftrightarrow P(\mathbf{a} + z\mathbf{v}) \in \mathcal{H}_1(\mathbb{R}) \quad \forall(\mathbf{a}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n_+.$$

$$\begin{aligned} \mathcal{H}_1(\mathbb{R}) &= \{P \in \mathbb{R}[z] : P^{-1}(0) \subset \mathbb{R}\} \\ &= \{\text{univariate hyperbolic polynomials}\}. \end{aligned}$$

Hence real-stability = natural multivariate extension of (univariate) hyperbolicity.

(Gårding) hyperbolicity. A homogeneous polynomial $P \in \mathbb{R}[z_1, \dots, z_n]$ is hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^n$ if $P(\mathbf{e}) \neq 0$ and $P(\mathbf{a} + z\mathbf{e}) \in \mathcal{H}_1(\mathbb{R})$ for all $\mathbf{a} \in \mathbb{R}^n$.

Lemma. Let $P \in \mathbb{R}[z_1, \dots, z_n]$ with $\deg P = d$ and let $\tilde{P}(z_0, z_1, \dots, z_n)$ be the (unique) homogeneous polynomial of degree d such that $\tilde{P}(1, z_1, \dots, z_n) = P(z_1, \dots, z_n)$. Then $P \in \mathcal{H}_n(\mathbb{R}) \Leftrightarrow \tilde{P}$ is hyperbolic w.r.t. every $\mathbf{e} \in \mathbb{R}^n$ s.t. $e_0 = 0$, $e_i > 0$, $1 \le i \le n$.

These are fundamental objects in many areas:

- PDE theory (Atiyah-Bott-Gårding, Lax, Hörmander)
- Singularity theory/hyperbolicity domains (Arnold)
- Matrix theory/optimization theory: Alexandrov-Fenchel type inequalities, Van der Waerden permanental and mixed discriminant conjectures (Egorychev, Gurvitz, Friedland); Lax conjecture for Gårding polynomials (Helton-Vinnikov, Lewis); Horn conjecture & related questions for Hermitian matrices (Klyachko, Knutson-Tao)
- Lee-Yang theory (Lieb-Sokal), graph combinatorics, matroid theory, electrical network theory (Barvinok, Choe-Oxley-Sokal-Wagner)
- Control theory/stability problems (Hurwitz), function theory, $\xi\left(\frac{1}{2}+it\right)$ (RH, Pólya-Schur, Levin, Craven, Csordas), orthogonal polynomials, etc.

Other names: "P-polynomials", "POS-polynomials", "HPP-polynomials", "wide sense stable polynomials"

Question. How to attack the classification problem?

Fact. Set $z = (z_1, \ldots, z_n)$. Any linear operator T on $\mathbb{C}[z_1, \ldots, z_n]$ may be uniquely represented as a (formal) differential operator with polynomial coefficients:

$$T = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta} z^\alpha \partial^\beta,$$

where $a_{\alpha\beta} \in \mathbb{C}$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. (Use induction or invoke Peetre's theorem.)

Let $w = (w_1, \ldots, w_n)$. The symbol of T is given by

$$F_T(z, w) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta} z^\alpha w^\beta \in \mathbb{C}[z_1, \ldots, z_n][[w_1, \ldots, w_n]].$$

The n-th Weyl algebra(s) $\mathcal{A}_n[\mathbb{C}]$ (respectively, $\mathcal{A}_n[\mathbb{R}]$) consists of all *finite order* linear differential operators with polynomial coefficients in $\mathbb{C}[z_1, \ldots, z_n]$ (respectively, $\mathbb{R}[z_1, \ldots, z_n]$).

First main objective: describe $\mathcal{A}_{RS}(n) \cap \mathcal{A}_n[\mathbb{R}]$ and $\mathcal{A}_S(n) \cap \mathcal{A}_n[\mathbb{C}]$.

Geometric description of n-RSPOs and n-SPOs

Theorem. Let $T \in \mathcal{A}_n[\mathbb{R}]$. Then $T \in \mathcal{A}_{RS}(n)$ if and only if $F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \in \mathcal{H}_{2n}(\mathbb{R})$.

So finite-order n-RSPOs are generated by real-stable polynomials in $2n$ variables via the symbol map.

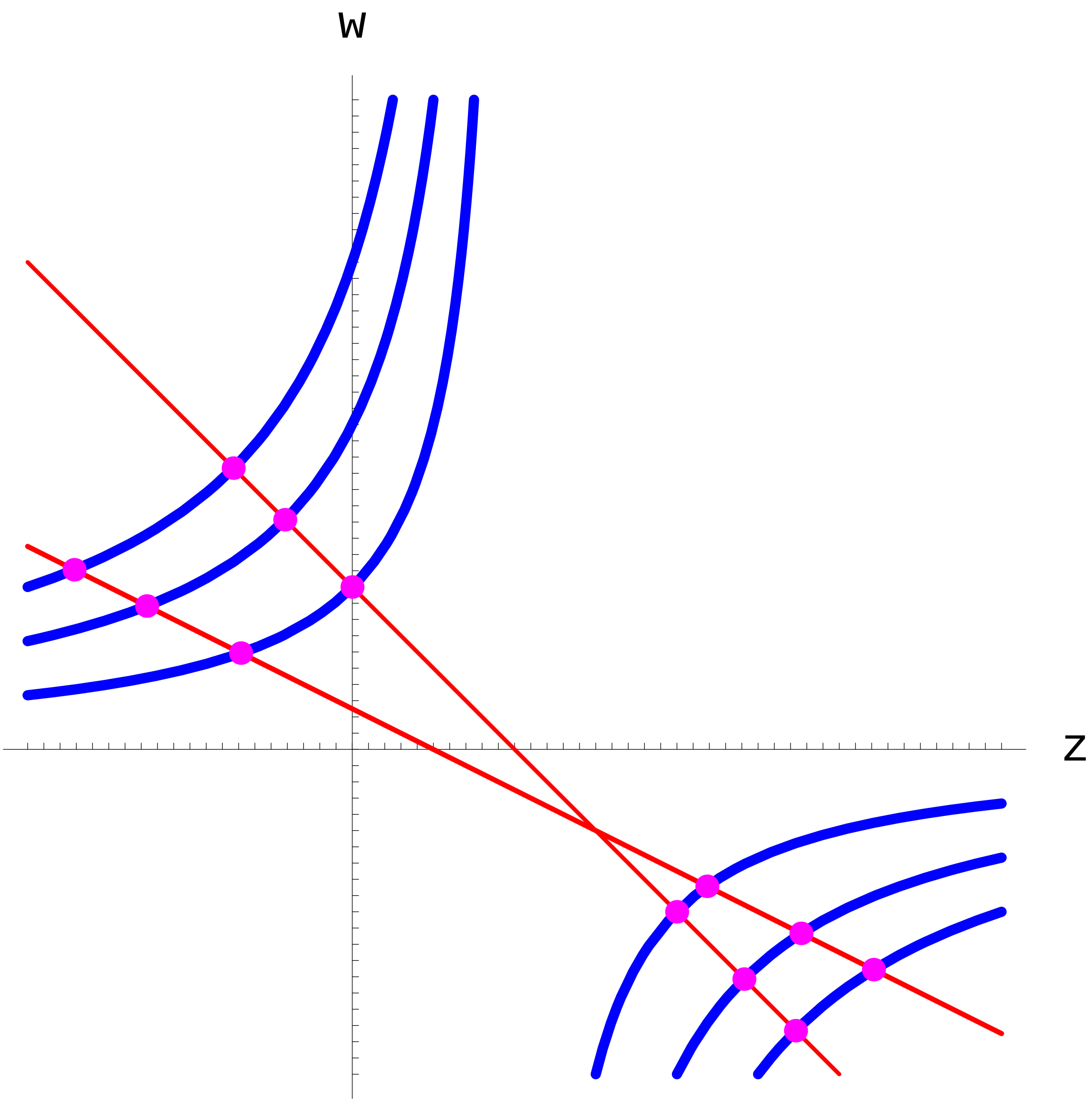
Visualizing RSPOs in the case $n = 1$:

Given $T \in \mathcal{A}_1[\mathbb{R}]$ with symbol $F_T(z, w) \in \mathbb{R}[z, w]$ of degree d consider the corresponding real algebraic (symbol) curve of degree d:

$$\mathbf{\Gamma_T = \{(z, w) \in \mathbb{R}^2 : F_T(z, w) = 0\}}.$$

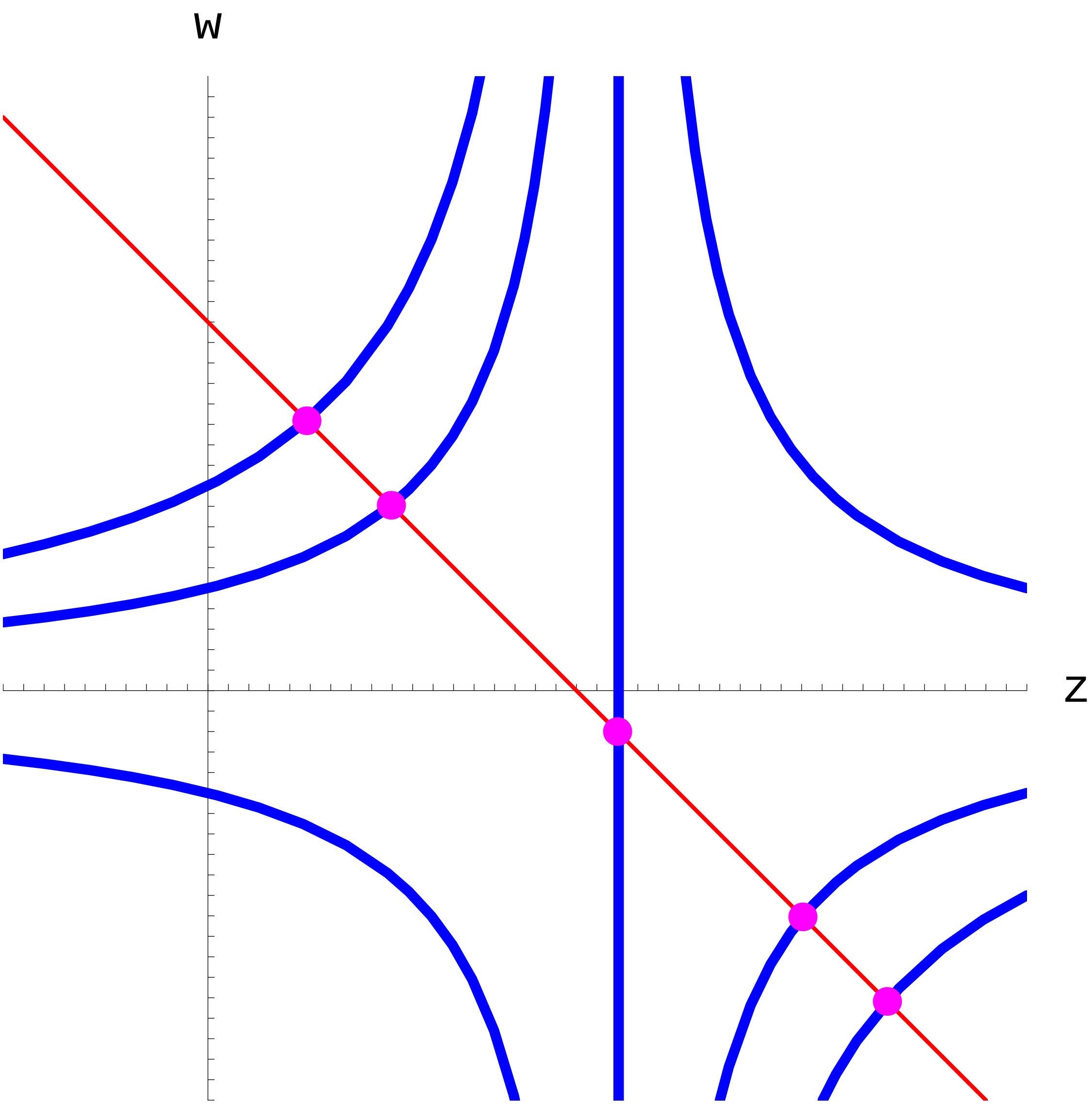
Geometric meaning of the above theorem: T is an RSPO if and only if each line $\mathcal{L}$ of negative slope intersects $\mathbf{\Gamma_T}$ in exactly $\mathbf{d}$ points (counting multiplicities).

Pictorially:



T is an RSPO $\Leftrightarrow |\mathcal{L} \cap \mathbf{\Gamma_T}| = \mathbf{d} \quad \forall \mathcal{L}$ as above.

Thus if $\mathbf{\Gamma_T}$ looks e.g. like this ($\mathbf{d} = 7$)



then $|\mathcal{L} \cap \mathbf{\Gamma_T}| = 5 < \mathbf{d}$ so T is not an RSPO!

Similarly, finite-order n-SPOs are generated by stable polynomials in $2n$ variables via the symbol map:

Theorem. Let $T \in \mathcal{A}_n[\mathbb{C}]$. Then $T \in \mathcal{A}_S(n)$ if and only if $F_T(z_1, \ldots, z_n, -w_1, \ldots, -w_n) \in \mathcal{H}_{2n}(\mathbb{C})$.

Some of the key ingredients in the proofs:

- Introduce multivariate multiplier sequences as "diagonal real-stability preservers", completely characterize them as well as all multivariate multiplier sequences in the n-th Weyl algebra $\mathcal{A}_n[\mathbb{R}]$.

- Establish homotopical properties of (R)SPOs: if $F(z, w)$ is the symbol of an n-(R)SPO then so is $F(\lambda z, \mu w)$ $\forall (\lambda, \mu) \in [0, 1]^n \times [0, 1]^n$.

- Multivariate Grace-Schur-Szegö-Walsh apolarity (*à la* Lieb-Sokal), "affine differential contractions" of symbols: if $P(y_1, \ldots, y_m) \in \mathcal{H}_m(\mathbb{C})$, $i < j$, then $P(y_1, \ldots, y_{i-1}, -\partial_j, y_{i+1}, \ldots, y_j, \ldots, y_m) \in \mathcal{H}_{m-1}(\mathbb{C}) \cup \{0\}$.

The Fischer-Fock space and a duality theorem

The Fischer-Fock (Bargmann-Segal) space $\mathcal{F}_n$ is the Hilbert space of holomorphic functions f on $\mathbb{C}^n$ s.t.

$$\pi^{-n}\int |f(z)|^2 e^{-|z|^2} dz_1 \wedge \cdots \wedge dz_n < \infty.$$

The inner product in $\mathcal{F}_n$ is given by

$$\langle f, g\rangle = \pi^{-n}\int f(z)\overline{g(z)} e^{-|z|^2} dz_1 \wedge \cdots \wedge dz_n.$$

Monomials $\{z^\alpha/\sqrt{\alpha!}\}_{\alpha\in\mathbb{N}^n}$ form an ON basis and

$$\langle \partial_i z^\alpha, z^\beta\rangle = \alpha!\delta_{\alpha=\beta+\mathrm{e}_i} = \langle z^\alpha, z_i z^\beta\rangle, \quad 1 \le i \le n.$$

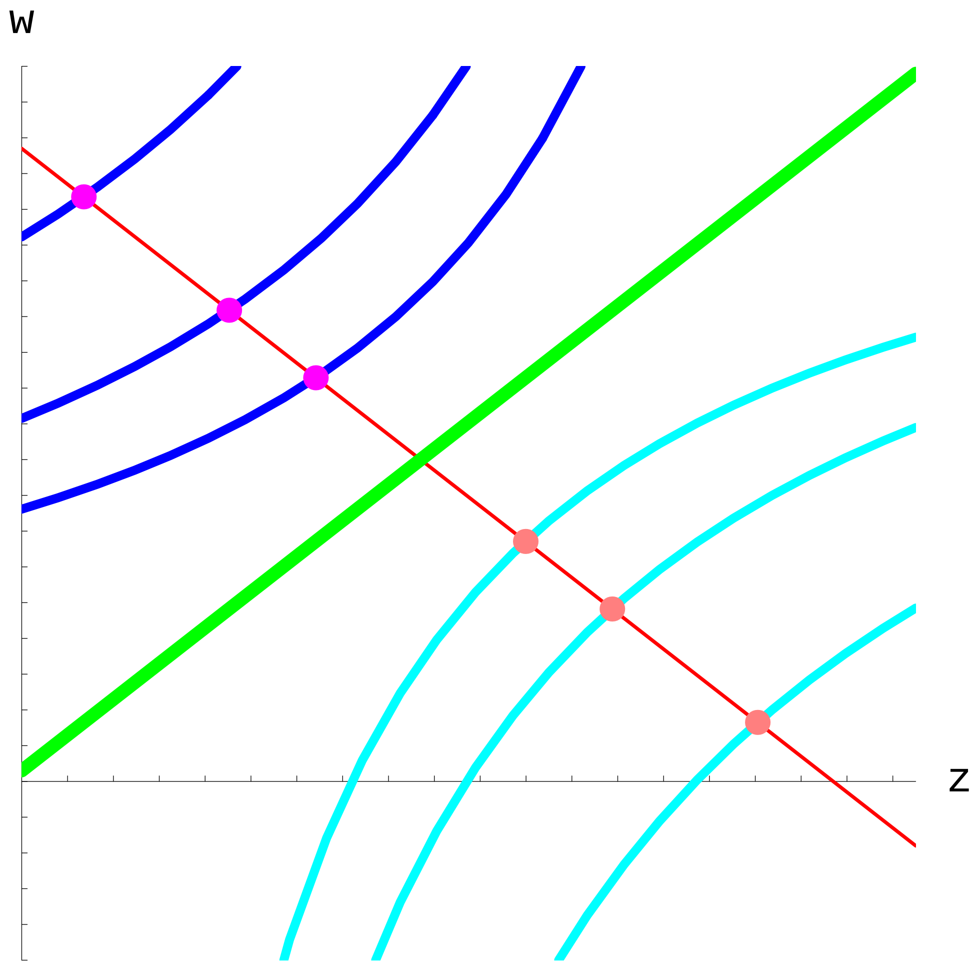
Hence if $T = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha \partial^\beta \in \mathcal{A}_n[\mathbb{C}]$ then

$$\begin{aligned}\langle T(f), g\rangle &= \sum_{\alpha,\beta} a_{\alpha\beta}\langle z^\alpha\partial^\beta f, g\rangle \\ &= \sum_{\alpha,\beta} a_{\alpha\beta}\langle f, z^\beta\partial^\alpha g\rangle = \langle f, \sum_{\alpha,\beta}\overline{a_{\beta\alpha}} z^\alpha\partial^\beta g\rangle\end{aligned}$$

so the (formal) adjoint of T is $T^* = \sum_{\alpha,\beta}\overline{a_{\beta\alpha}} z^\alpha\partial^\beta$ and $F_{T^*}(z,w) = \overline{F_T(\overline{w},\overline{z})}$.

Duality theorem. If $T \in \mathcal{A}_n[\mathbb{C}]$ (resp., $\mathcal{A}_n[\mathbb{R}]$) then $T \in \mathcal{A}_S(n) \quad (\mathcal{A}_{RS}(n)) \Leftrightarrow T^* \in \mathcal{A}_S(n) \quad (\mathcal{A}_{RS}(n))$.

“Drawing duality” for $n = 1$: $F_{T^*}(z, w) = F_T(w, z)$ if $T \in \mathcal{A}_1[\mathbb{R}]$ so $\mathbf{\Gamma_{T^*}}$ is the reflection of $\mathbf{\Gamma_T}$ in $\mathbf{z = w}$. The “intersection property” is preserved by reflection in $\mathbf{z = w}$ $\Rightarrow$ duality for $\mathcal{A}_{RS}(1) \cap A_1[\mathbb{R}]$.



Vastly generalizes the Hermite-Poulain-Jensen theorem (if $T = P(D)$ with $P \in \mathbb{R}[z]$ then $T \in \mathcal{A}_{RS}(1)$ $\Leftrightarrow P \in \mathcal{H}_1(\mathbb{R})$). Gives natural multivariate extension.

Some applications

- Determinantal characterization of 1-RSPOs

Lax conjectured (1950's), Lewis *et al* proved (2005):

Theorem. A homogeneous polynomial $P \in \mathbb{R}[x, y, z]$ of degree d is hyperbolic w.r.t. $\mathbf{e} = (1, 0, 0)$ if and only if there exist symmetric $d \times d$ matrices A, B such that $P(x, y, z) = P(e)\,\text{det}(xI + yA + zB)$.

"Lax conjecture" for $\mathcal{H}_2(\mathbb{R})$:

Theorem. Let $P \in \mathbb{R}[z, w]$. Then $P \in \mathcal{H}_2(\mathbb{R})$ if and only if there exist positive semidefinite (PSD) $d \times d$ matrices A, B and a symmetric $d \times d$ matrix C such that $P(z, w) = \alpha\,\text{det}(C + zA + wB)$, where $\alpha \in \mathbb{R}$.

Corollary. Let $T \in \mathcal{A}_1[\mathbb{R}]$. Then $T \in \mathcal{A}_{RS}(1)$ if and only if there exist PSD $d \times d$ matrices A, B and a symmetric $d \times d$ matrix C such that

$$T = \alpha\,\text{det}(C + zA - wB)\Big|_{w=\frac{\partial}{\partial z}}, \quad \alpha \in \mathbb{R}.$$

- Description of 1-RSPOs via homogenized symbols

Theorem. Let $T \in \mathcal{A}_1[\mathbb{R}]$ with symbol $F_T(z,w)$ of degree d and let $\tilde{F}_T(y,z,w)$ be the (unique) homogeneous degree d polynomial such that $\tilde{F}_T(1,z,w) = F_T(z,w)$. Then $T \in \mathcal{A}_{RS}(1)$ if and only if:

(i) $\tilde{F}_T(y,z,w)$ is hyperbolic w.r.t. $(0,1,1)$ and

(ii) all zeros of $\tilde{F}_T(0,t,1)$ lie in $(-\infty,0]$.

- Examples of real-stable polynomials

Theorem. Let A_i, $1 \leq i \leq n$, be PSD $m \times m$ matrices and B be a Hermitian $m \times m$ matrix. Set

$$f(z_1,\ldots,z_n) = \det\left(\sum_{i=1}^{n} z_i A_i + B\right) \in \mathbb{R}[z_1,\ldots,z_n].$$

Then $f \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$.

- Multivariate extensions of the characteristic polynomial and the Poincaré-Cauchy interlacing theorem

Given an $n \times n$ matrix A let

$$\chi(A, \mathbf{z}) := \det(Z - A) \in \mathbb{C}[z_1, \ldots, z_n],$$

where $Z = (z_i \delta_{ij})$. Let A^{ij} be the submatrix of A obtained by removing row i and column j and set

$$\chi_{ij}(A, \mathbf{z}) = \det((Z - A)^{ij}).$$

If $\mathbf{z} = (z_1, \ldots, z_n)$ and $1 \leq i \leq n$ let

$$\mathbf{z} \setminus z_i := (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$$

Note that $\chi_{ii}(A, \mathbf{z}) = \chi(A^{ii}, \mathbf{z} \setminus z_i) = \frac{\partial}{\partial z_i}\chi(A, \mathbf{z})$.

Definition. Two polynomials $f, g \in \mathcal{H}_n(\mathbb{R})$ are in proper position, denoted $f \ll g$, if $g + if \in \mathcal{H}_n(\mathbb{C})$.

This turns out to be an appropriate multivariate extension of interlacing in view of the Hermite-Biehler theorem and Obreschkoff's theorem.

Theorem. For any Hermitian $n \times n$ matrix A one has $\chi(A, \mathbf{z}) \in \mathcal{H}_n(\mathbb{R})$ and $\chi(A^{jj}, \mathbf{z} \setminus z_j) \ll \chi(A, \mathbf{z})$ $\forall j$.

- Inertia laws for mixed determinants

Given an $n \times n$ matrix A and $\alpha \subseteq \{1,\ldots,n\}$ let $A[\alpha]$ be the principal submatrix lying in the rows and columns indicated by α. Denote by $A(\alpha)$ the complementary principal submatrix, so that $A(\alpha) = A[\alpha']$. For $n \times n$ matrices A, B and $0 \le k \le n$ let

$$S_k(A,B) := \sum_{|\alpha|=k} \det A[\alpha] \det B(\alpha),$$

$$\eta_{A,B}(z) := \sum_{k=0}^{n} (-1)^k S_k(A,B) z^k \in \mathbb{C}[z].$$

Clearly, $\eta_{I,B}(z) = \det(zI - B)$. Johnson & Bapat conjectured the following (~ 1987):

Theorem. If A, B are Hermitian and A is PSD then

(i) $\eta_{A,B}(z)$ has all real roots,

(ii) $\eta_{A,B}(z)$ has as many positive, negative and zero roots (counting multiplicities) as the inertia of B,

(iii) $\eta_{A(\{i\}),B(\{(i)\})}(z)$ interlaces $\eta_{A,B}(z)$, $1 \le i \le n$.

Corollary (Barrett-Johnson, 1993). If A is PSD then its average Fischer terms $\binom{n}{k}^{-1} S_k(A,A)$ are monotone increasing from $k = 0$ to $k = \left[\frac{n}{2}\right]$.

Statement (i) in the above theorem may be generalized to $\eta_{A,B_1,\ldots,B_m}(z)$, where $m \le n$, A is PSD and B_i, $1 \le i \le m$, are Hermitian $n \times n$ matrices.

- Laguerre-type inequalities for Hermitian matrices

Laguerre's inequality. If $p(z) \in \mathbb{R}[z]$ has all real roots then $p'(z)^2 \ge p''(z)p(z)$ $\forall z \in \mathbb{R}$.

Theorem. If A is a Hermitian $n \times n$ matrix then

$$\begin{aligned}\det(zI_{n-1} - A(\{i\}))\det(zI_{n-1} - A(\{j\})) \\ \ge \det(zI_{n-2} - A(\{ij\}))\det(zI_n - A)\end{aligned}$$

$\forall z \in \mathbb{R}$ and $1 \le i \ne j \le n$. In particular,

$$\det A(\{i\}) \det A(\{j\}) \ge \det A(\{ij\}) \det A.$$

Corollary (Mirsky, 1963). If A is a positive definite $n \times n$ matrix and $1 \le i \ne j \le n$ then

$$\frac{\det A}{\det A(\{i\})} \le \frac{\det A(\{j\})}{\det A(\{ij\})}.$$

In particular, $\det A \le \prod_{i=1}^{n} a_{ii}$ (Hadamard).

Towards a complete classification of 1-RSPOs

Definition. A linear operator T on $\mathbb{C}[z]$ is monotone if there exist $M \in \mathbb{N}$ and $d \in \mathbb{Z}$ such that

$$T(z^n) = 0,\ n < M,\ \ \deg T(z^n) = n + d,\ n \geq M.$$

$d =$ degree shift of T. Such a T is sign-monotone if either the leading coefficient of $T(z^n)$ is $\geq 0\,\forall n \in \mathbb{N}$ (T is positive-monotone) or the leading coefficient of $T(z^n)$ is $\leq 0\,\forall n \in \mathbb{N}$ (T is negative-monotone). Clearly, T is negative-monotone if and only if $-T$ is positive-monotone. Let $\nu : \mathbb{C}[z] \to \mathbb{C}[z]$ be the linear involution defined by $\nu(f)(z) = f(-z)$. T is called alternating-monotone if $T \circ \nu$ is sign-monotone.

Theorem. A monotone 1-RSPO is either sign-monotone or alternating-monotone.

Conjecture.* If $T \in \mathcal{A}_{RS}(1)$ then T is either monotone or it has finite-dimensional range.

*Now a 2-days old theorem...

Theorem. Let $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ be a positive-monotone linear operator on $\mathbb{C}[z]$, where we assume that the Q_k's have no common non-trivial divisor. Set

$$d = \max\{i : \exists \theta \in \mathbb{R} \text{ s.t. } Q_0(\theta) = Q_1(\theta) = \dots = Q_{i-1}(\theta) = 0\}.$$

Then $T \in \mathcal{A}_{RS}(1)$ if and only if

(i) $F_T(z, w) \in \mathcal{LP}(w)$ for each fixed $z \in \mathbb{R}$ and

(ii) the zeros of $T(P')$ and $T(P)$ interlace, where e.g. $P(z) = \prod_{k=1}^{d}(z - k)$.

Corollary. Description of the following classes:

(i) 1-RSPOs commuting with D (Pólya-Benz, 1934),

(ii) unipotent upper triangular 1-RSPOs (Pinkus *et al*, 2003), etc.