Inverse Problems for Quantum Trees

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Let $\Omega \subset \mathbf{R}^2$ be a finite connected planar graph without cycles with edges $\{e\} = E$ (intervals of straight lines), vertices $\{v_1, \ldots, v_m\} = V$ and boundary vertices $\{\gamma_0, \gamma_1, \ldots, \gamma_N\} = \Gamma \subset V$.

To the graph we associate the boundary value problem

$$-\frac{d^2\psi}{de^2} + q\,\psi = \lambda\psi \quad \text{in } \{\Omega \setminus V\} \tag{1}$$

$$\psi \in C(\bar{\Omega})\,; \quad \sum_{e\sim v} \frac{d\psi}{de}(v) = 0 \quad \text{for all } v \in V \setminus \Gamma \tag{2}$$

$$q \in L^1(\Omega)$$

To define the Weyl matrix function let us denote by ψ_i, $i = 1, \ldots, N$, the solution of (1), (2) with boundary conditions

$$\psi_i(\gamma_i) = 1\,, \quad \psi_i = 0 \quad \text{in } \Gamma \setminus \{\gamma_i\} \tag{3}$$

and introduce $N \times N$ matrix $M(\lambda)$ with entrees

$$m_{ij}(\lambda) = \frac{d\psi_i}{de}(\gamma_j)\,. \tag{4}$$

Theorem 1. *Matrix function $M(\lambda)$ uniquely determines graph Ω, this means that the Weyl function uniquely determines the graph's topology, the lengths of the edges and the potentials on them.*

Theorem 2. *If the topology of the graph is known, the lengths of the edges and potentials on them are uniquely determined by the diagonal terms* $m_{ii}(\lambda), i = 1, \ldots, N$, *of the Weyl matrix function (the so-called Weyl vector).*

Our proofs are constructive; together with the uniqueness theorems we give the constructive procedures of the solution of inverse problems.

Previous close results: Brown and Weikard (2005), Yurko (2005), Belishev (2004).

We consider also the dynamical system

$$u_{tt} - u_{ee} + q\,u = 0 \quad \text{in } \{\Omega \setminus V\} \times (0,T)\,;$$

$$\sum_{e \sim v} u_e\,|_{x=v} = 0 \quad \text{for} \quad \text{all } v \in V \setminus \Gamma\,;$$

$$u\,|_{t=0} = u_t\,|_{t=0} = 0 \quad \text{in } \Omega\,;$$

$$u(\gamma_0, t) = 0, \quad u = f \quad \text{on } \Gamma_0 \times [0,T]\,,\ \Gamma_0 := \Gamma \setminus \{\gamma_0\}.$$

Let $u = u^f(x,t)$ be the solution of this initial boundary value problem.

By R^T we denote the response operator:

$$R^T f := \left.\frac{du^f}{de}\right|_{\Gamma_1 \times [0,T]}.$$

Connections between spectral and dynamical data

Let $f \in C_0^\infty(0,\infty;\mathbf{R}^N)$ and $F(k) := \int_0^\infty f(t)\, e^{ikt}\, dt\,.$

If $\Im k > 0$, then for any $p > 0$

$$|F(k)| \leq C_p(1+|k|)^{-p} \tag{5}$$

Let ψ be a solution of the equation

$$-\psi(x,k) + q(x)\psi(x,k) = k^2\psi(k,x) \tag{6}$$

with boundary conditions

$$\psi(\cdot,k) = F(k) \quad \text{on } \Gamma_0\,, \quad \psi(\gamma_0,k) = 0\,.$$

Estimate (5) implies that $|\psi(x,k)|$ decreases rapidly when $|k| \to \infty\,,\ \Im k \geq \varepsilon > 0\,.$ Moreover,

$$\psi'(\cdot,k) = M(k^2)F(k)\,. \tag{7}$$

The Fourier transform

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,\kappa+i\nu)\, e^{-i(\kappa+i\nu)t}\, d\kappa\,,\ \ \nu>0\,,$$

defines the function which solves the initial boundary value problem.

The Weyl matrix function $M(\lambda)$ determines the response operator R^T for all $T>0$:

$$R^T f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M\left((\kappa+i\nu)^2\right)\, F(\kappa+i\nu)\, e^{-i(\kappa+i\nu)t}\, d\kappa\,,\ t\in[0,T]\,. \tag{8}$$

Theorem 3. Let $T>T_*$. (a) *Operator* R^T *uniquely determines the graph's topology, the lengths of the edges and the potentials on them.* (b) *If the topology of the graph is known, the lengths of the edges and potentials on them are uniquely determined by the diagonal terms* $(R^T)_{ii}, i=1,\dots,N,$.

The Boundary Control (BC) method

$$w_{tt}(x,t) - w_{xx}(x,t) + q(x)w(x,t) = 0, \quad x \in (0,\ell),\ t \in (0,T), \tag{9}$$

with the boundary conditions

$$w(0,t) = f(t), \quad w(\ell,t) = 0,\ t \in (0,T), \tag{10}$$

and zero initial conditions

$$w(x,0) = w_t(x,0) = 0. \tag{11}$$

Proposition 1. (i) If $f \in C^2[0,T]$ and $f(0) = f'(0) = 0$, the problem (9)–(11) has the unique solution $w = w^f(x,t)$,

$$w^f(x,t) = \begin{cases} f(t-x) + \int_x^t h(x,s) f(t-s)\, ds & \text{for } x < t \\ 0 & \text{for } x \geq t\,, \end{cases} \tag{12}$$

$w^f \in H^2(Q^T)$, $Q^T := (0,\ell) \times (0,T)$, equation (9) is satisfied almost everywhere, and the boundary and initial conditions are satisfied in a classical sense.

(ii) For $f \in L^2(0,T) := \mathcal{F}^T$ the function $w^f(x,t)$ defined by (12) gives a generalized solution of problem (9)–(11) such that $w^f \in C([0,T]; L^2(0,\ell))$.

Here h is a solution of the Goursat problem:

$$h_{tt} - h_{xx} + q(x)h = 0, \quad 0 < x < t < T\,, \tag{13}$$

$$h(0,t) = 0\,, \quad h(x,x) = -\frac{1}{2}\int_0^x q(s)\,ds\,. \tag{14}$$

The **response operator** $R^T: \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$(R^T f)(t) = w_x^f(0,t),\ Dom\ R^T = \{f \in C^2[0,T]:\ f(0) = f'(0) = 0\}\,. \tag{15}$$

By the force of (12) for $T \leq 2\ell$ we have a representation

$$(R^T f)(t) = -f'(t) + \int_0^t r(\tau) f(t-\tau)\,d\tau\,, \quad t \in [0,T]\,. \tag{16}$$

The **control operator** $\mathcal{W}^T$,

$$\mathcal{W}^T : \mathcal{F}^T \mapsto \mathcal{H}^T, \ \mathcal{W}^T f = w^f(\cdot, T),$$

is bounded. Here $\mathcal{H}^T := \{u \in L^2(0, \ell) : \text{ supp } u \subset [0, T]\}$.

We can prove a controllability result which shows that the operator $\mathcal{W}^T$ is boundedly invertible:

Proposition 2. For any function $z \in \mathcal{H}^T$, there exists a unique control $f \in \mathcal{F}^T$ such that

$$w^f(x, T) = z(x) \text{ in } \mathcal{H}^T. \tag{17}$$

Now we show how to recover the potential function $q(x)$ from the known $R^T, \ T \geq 2\ell$.

The **connecting operator** $\mathcal{C}^T : \mathcal{F}^T \to \mathcal{F}^T$:

$$\left(\mathcal{C}^T f, g\right)_{\mathcal{F}^T} = \left(w^f(\cdot, T), w^g(\cdot, T)\right)_{\mathcal{H}}.$$

The operator $\mathcal{C}^T$ is bounded and boundedly invertible, since $\mathcal{C}^T = \left(\mathcal{W}^T\right)^* \mathcal{W}^T$.

Operator $\mathcal{C}^T$ plays a central role in the BC method.

$$(\mathcal{C}^T f)(t) = f(t) + \int_0^T [p(2T - t - s) - p(|t - s|)] f(s)\, ds\,, \qquad (18)$$

where

$$p(t) = \frac{1}{2} \int_0^t r(s)\, ds\,.$$

The Gel'fand–Levitan equation

Let $y(x)$ be a solution to the boundary value problem

$$y''(x) - q(x)y(x) = 0,\ y(0) = 0,\ y'(0) = 1, x \in (0,\ \ell) \tag{19}$$

and let us find a control $f^T \in \mathcal{F}^T$ such that

$$w^{f^T}(x, T) = \begin{cases} y(x), & x \le T, \\ 0, & x > T. \end{cases} \tag{20}$$

the function f^T satisfies the equation

$$\left(\mathcal{C}^T f^T\right)(t) = T - t, \quad t \in [0, T].$$

Since $\mathcal{C}^T$ is boundedly invertible this equation has a unique solution, $f^T \in \mathcal{F}^T$, for any $T \le \ell$.

Moreover, it can be proved that $f^T \in H^1(0, T)$.

For any $g \in C_0^\infty[0,T]$, we have

$$(\mathcal{C}^T f^T, g)_{\mathcal{F}^T} = \Big(w^{f^T}(\cdot,T), w^g(\cdot,T)\Big)_{\mathcal{H}}$$

$$= \int_0^T y(x) w^g(x,T)\,dx = \int_0^T (T-t)\,dt \int_0^T y(x) w^g_{tt}(x,t)\,dx$$

$$= \int_0^T (T-t)\,dt \int_0^T y(x)\left[w^g_{xx}(x,t) - q(x) w^g(x,t)\right]\,dx$$

$$= \int_0^T (T-t)[(y(x)w^g_x(x,t) - y'(x)w^g(x,t)]|_{x=0}^T\,dt$$

$$= \int_0^T (T-t) g(t)\,dt$$

(we used that for $g \in C_0^\infty[0,T]$, the function w^g and its derivatives are equal to zero at $x = T$).

Applying the propagation of singularities property, we obtain

$$w^{f^T}(T-0,T) = f^T(+0) := \mu(T)\,.$$

From (20), $w^{f^T}(T-0,T) = y(T)$, thus (19) gives

$$q(T) = \frac{y''(T)}{y(T)} = \frac{\mu''(T)}{\mu(T)}.$$

By varying T in $(0,\ell)$, we obtain $q(\cdot)$ in that interval. Since the function $y(T)$ may have only a finite number of zeroes in $(0,\ell)$, this completes the solution of the identification problem.

For $2\ell < T \leq 3\ell$ we have a representation

$$(R^T f)(t) = -f'(t) - 2f'(t)(t-2\ell) + \int_0^t r(t,\tau)f(\tau)\,d\tau\,, \quad t \in [0,T] \tag{21}$$

which allows us to find ℓ.

To solve inverse problem for a graph we use the following relations

$$(R_{11}^T f)(t) = -f'(t) - 2\frac{n-2}{n} f'(t-2l_1) + \int_0^t r(t,\tau) f(\tau) d\tau \quad (22)$$

$$t \in [2l_1, 2l_1 + \min \{l_j\}_{j=1}^n] \, .$$

$$R_{ij}^T = \begin{cases} = 0, & \text{for}\, T < l_i + l_j; \\ \neq 0, & \text{for}\, T > l_i + l_j; \end{cases}$$