On the Inverse Spectral Problem for Graphs with Cycles

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Quantum graph as a triplet

1. **Metric graph** $\Gamma$ - union of intervals $\Delta_j = [x_{2j-1}, x_{2j}]$ connected together at the vertices $V_m$ considered as equivalence classes of end-points $\Rightarrow$ the Hilbert space $L_2(\Gamma)$;

2. **Differential expression** (formally symmetric) on the edges

$$L_{q,a} = \left( -\frac{1}{i} \frac{d}{dx} + a(x) \right)^2 + q(x)$$

$\Rightarrow$ the linear operator $L_{q,a}$;

3. **Boundary conditions** at the vertices
   - to determine $L_{q,a}$ as a self-adjoint operator,
   - connect together different edges.

In this talk we are going to speak only about the **standard boundary conditions** only, that is:
   - the function is continuous,
   - the sum of ”normal” derivatives is zero.
Quantum graph as a triplet

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"Elimination" of the magnetic field

Consider the unitary transformation:

\[(U\psi)(x) = \exp \left( -i \int_{x_{2n-1}}^{x} a(y) dy \right) \psi(x), \quad x \in (x_{2n-1}, x_{2n}), \quad n = 1, 2, ..., N,\]

which allows one to eliminate the magnetic field

\[U \left( \left( -\frac{1}{i} \frac{d}{dx} + a(x) \right)^2 + q(x) \right) U^{-1} \psi(x) = -\frac{d^2}{dx^2} \psi(x) + q(x) \psi(x).\]

**NB!** The magnetic field can be eliminated from the differential expression, but then it appears in the boundary conditions (if the graph is not a tree).

**Proposition 1.** The spectrum of the magnetic Schrödinger operator \(L_{q,a}\) is pure discrete and does not depend on the particular form of the magnetic field but just on the fluxes of the magnetic field through the cycles

\[\Phi_j = \int_{c_j} a(y) dy.\]
Inverse problems: concise historical overview

Solution of the inverse problem for quantum graphs means reconstruction of

- the metric graph;
- the differential expressions on the edges;
- the coupling conditions at the vertices.

Obtained results NB! for zero magnetic potential!

- Reconstruction of the graph:
  - with rationally independent lengths:
    B. Gutkin, T. Kottos and U. Smilansky, '99, '01;
    P. K., F. Stenberg and M. Nowaczyk '02, '05, '07, '08;
  - in the case of tree:
    V. Yurko '06,
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  - calculation of the Euler characteristic:
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Reconstruction of the potential on graphs:
- star graph:
  N.I. Gerasimenko and B.S. Pavlov, 1988;
- tree:
  M. Belishev and A. Vakulenko, ’04, ’06, ’07;
  M. Brown and R. Weikard, ’05;
  V. Yurko ’05, ’06, ’08;
  S. Avdonin and P. K. ’08;
- impossibility for loops:
  J. Boman and P. K., 05
  V. Pivovarchik, manuscript;

Reconstruction of the boundary conditions:
- for star graphs:
  V. Kostrykin and R. Schrader ’00, ’06;
  M. Harmer ’03.

Other references:
General overview: P. Kuchment ’04.
The main idea

**Conclusions** concerning recovering the potential

- Knowledge of the spectrum alone is not enough to reconstruct the potential.
- Titchmarsh-Weyl function (equivalently the Dirichlet-to-Neumann map) is an efficient tool to solve the inverse problem for graphs.
- Potential on the branches can be reconstructed from the TW function using Boundary Control method.
- Potential on the kernel of the graph in general cannot be determined by the TW function.

**Our programme**

Study the possibility to reconstruct the graph $\Gamma$ and potential $q$ on it from the TW function known for different values of the magnetic field.
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Marchenko-Ostrovsky theory

Provides necessary and sufficient conditions for a sequence of intervals to be the spectrum of one-dimensional periodic Schrödinger operator $L^\text{per}_q$.

Transfer matrix $T(a, b; \lambda)$

$$-\frac{d^2}{dx^2} \psi(x) + q(x) \psi(x) = \lambda \psi(x) \Rightarrow T(a, b; \lambda) : \begin{pmatrix} \psi(a) \\ \psi'(a) \end{pmatrix} \mapsto \begin{pmatrix} \psi(b) \\ \psi'(b) \end{pmatrix}$$

Introduce the functions:

$$u_\pm(\lambda) = \left(t_{11}(\lambda) \pm t_{22}(\lambda)\right)/2$$

The end points of the spectral intervals $\mu_j, \tilde{\mu}_j$ are solutions to the equation

$$u_\pm(\lambda) = \pm 1.$$
Proposition 2. For the sequences

\[ 0 = \tilde{\mu}_0 < \mu_1 \leq \tilde{\mu}_1 < \mu_2 \leq \tilde{\mu}_2 < \ldots \]  

(1)

to be the spectra of periodic and antiperiodic boundary value problems generated on the interval \([0, \pi]\) by the operator \(-\frac{d^2}{dx^2} + q(x)\) with real potential \(q(x) \in W_2^n[0, \pi]\), it is necessary and sufficient that there exist a sequence of real numbers \(h_k (k = 0, \pm 1, \pm 2, \ldots)\) satisfying the conditions

\[ \sum_{k=1}^{\infty} (k^{n+1} h_k)^2 < \infty, \quad h_0 = 0, \quad h_k = h_{-k} \geq 0 (k = 1, 2, \ldots), \]  

(2)

such that

\[ \sqrt{\mu_k} = z(\pi k - 0), \quad \sqrt{\tilde{\mu}_k} = z(\pi k + 0) \quad (k = 1, 2, \ldots), \]

where the function \(z(\theta)\) effects a conformal mapping of the region

\[ \{\theta : \text{Im} \theta > 0\} \setminus \bigcup_{k=-\infty}^{+\infty} \{\theta : \text{Re} \theta = k\pi, 0 \leq \text{Im} \theta \leq h_k\} \]  

(3)

into the upper half-plane, normalized by the conditions

\[ \theta(0) = 0, \quad \lim_{y \to \infty} (iy)^{-1} \theta(iy) = \pi. \]
In fact the following statement has been proven in


\textbf{Proposition 3.} Assume that all conditions of Proposition 2 are satisfied. The following set of spectral data determine the potential uniquely:

- the spectrum of the periodic operator
  
  \[ [0 = \tilde{\mu}_0, \mu_1] \cup [\tilde{\mu}_1, \mu_2] \cup [\tilde{\mu}_2, \mu_3] \cup \ldots, \]

- the D-D spectrum \( \lambda_k \) satisfying \( \mu_j \leq \lambda_j \leq \tilde{\mu}_j \),

- the sequence of signs \( \nu_k = \pm 1 \).
Motivation for the Proposition

- In order to determine the potential it is enough to know the spectra of the DD and DN problems (Borg-Levitan-Marchenko). Equivalently it is enough to know the functions
  - $t_{22}(\lambda)$ - its zeroes form the spectrum of D-N problem;
  - $t_{12}(\lambda)$ - its zeroes form the spectrum of D-D problem.

- The spectrum of the periodic Schrödinger operator (periodic and antiperiodic problems) allows one to determine the quasimomentum $\theta(\lambda)$ so that we have
  \[
  u_+(\lambda) = \cos \theta(\lambda).
  \]

- The numbers $\lambda_k$ give the spectrum the D-D problem, or the function $t_{12}(\lambda)$.

- For $\lambda = \lambda_k$ we have:
  \[
  t_{11} + t_{22} = 2 \cos \theta(\lambda_k), \quad t_{11} t_{22} = 1 \Rightarrow u_-(\lambda_k) = \nu_k \sqrt{u_+^2 - 1}, \quad \nu_k = \pm 1.
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So in order to determine the D-N spectrum (the function $t_{22}$) one needs to know the sequence of signs $\nu_k$. 

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Motivation for the Proposition

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- The spectrum of the periodic Schrödinger operator (periodic and antiperiodic problems) allows one to determine the quasimomentum $\theta(\lambda)$ so that we have
  \[ u_{\pm}(\lambda) = \cos \theta(\lambda). \]

- The numbers $\lambda_k$ give the spectrum the D-D problem, or the function $t_{12}(\lambda)$.

- For $\lambda = \lambda_k$ we have:
  \[ t_{11} + t_{22} = 2 \cos \theta(\lambda_k), \quad t_{11} t_{22} = 1 \Rightarrow u_{-}(\lambda_k) = \nu_k \sqrt{u_{+}^2 - 1}, \quad \nu_k = \pm 1. \]

So in order to determine the D-N spectrum (the function $t_{22}$) one needs to know the sequence of signs $\nu_k$. 
The potential is uniquely determined by

- the function $u_+(\lambda)$;
- the function $t_{12}(\lambda)$;
- the function $u_-(\lambda)$. 
Inverse problems for simple graphs

Ring graph $\Gamma_1$

$\Phi_1$ - the total flux through the ring $\Phi_1 = \int_{x_1}^{x_2} a(y)dy$.

$L_{q,\Phi_1}$ - magnetic Schrödinger operator.

$E$ is an eigenvalue of $L_{q,\Phi_1}$ if and only if it belongs to the interval of the absolutely continuous spectrum of the periodic operator $L_{q,\text{per}}$ corresponding to the quasimomentum $\theta = \Phi_1$.

The knowledge of $E_n(\Phi_1)$ allows one to recover just

- the function $u_+(\lambda) = \text{Tr} \ T(\lambda)/2$.

The potential can be reconstructed only in the very exceptional case of zero or constant potential.
Lassoo graph $\Gamma_2$

The knowledge of the TW function $M_{\Phi_j}(\lambda, \Gamma_2)$ by Boundary-Control method allows one to determine the TW function $M_{\Phi_1}(\lambda, \Gamma_1)$ (where $\Gamma_1$ is the ring graph with one contact point)

$$M_{\Phi_1}(\lambda, \Gamma_1) = \frac{2 \cos \Phi_1 - \text{Tr} \ T(\lambda)}{t_{12}(\lambda)}.$$ 

The knowledge of the TW matrix for the magnetic flux $\Phi_1 = 0, \pi$ (and for all other values of $\Phi_1$) allows one to recover just

- the function $u_+(\lambda) = \text{Tr} \ T(\lambda)/2$;
- the function $t_{12}(\lambda)$.

To reconstruct the potential on the ring we need to know in addition the sequence of signs $\nu_k$ or, equivalently, the function $u_-(\lambda)$. Reconstruction of the potential on the ring can be carried out, but it is not unique. The potential on the boundary edge is uniquely determined by $M_{\Phi}(\lambda, \Gamma_2)$. 

Zweihänder graph $\Gamma_3$

The knowledge of the TW function $M_{\Phi_j}(\lambda, \Gamma_2)$ by the Boundary-Control method allows one to determine the $2 \times 2$ TW function $M_{\Phi_1}(\lambda, \Gamma_4)$, where $\Gamma_4$ is the ring graph with two contact points

$$
M(\lambda, \Gamma_4) = \frac{1}{t_{12}^1 t_{12}^2} \begin{pmatrix}
-(T^1 T^2)_{12} & t_{12}^1 e^{i\Phi_2} + t_{12}^2 e^{-i\Phi_1} \\
t_{12}^2 e^{i\Phi_1} + t_{12}^1 e^{-i\Phi_2} & -(T^2 T^1)_{12}
\end{pmatrix},
$$

where $T^{1,2}$ are the transfer matrices for the two intervals forming the circle.

NB! The TW matrix can be reconstructed up to the similarity transformation with diagonal unitary matrix

$$
M(\lambda) = \begin{pmatrix}
e^{i\Phi_3} & 0 \\
0 & e^{i\Phi_4}
\end{pmatrix} M(\lambda, \Gamma_4) \begin{pmatrix}
e^{-i\Phi_3} & 0 \\
0 & e^{-i\Phi_4}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-(T^1 T^2)_{12} t_{12}^1 t_{12}^2 & \frac{1}{t_{12}^1} + \frac{1}{t_{12}^2} e^{-i\Phi} e^{i(\Phi_2+\Phi_3-\Phi_4)} \\
(\frac{1}{t_{12}^1} + \frac{1}{t_{12}^2} e^{i\Phi}) e^{-i(\Phi_2+\Phi_3-\Phi_4)} & -(T^2 T^1)_{12} t_{12}^1 t_{12}^2
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$$M(\lambda, \Gamma_4) = \frac{1}{t_{12}^1 t_{12}^2} \begin{pmatrix} - (T^1 T^2)_{12} & t_{12}^1 e^{\Phi_2} + t_{12}^2 e^{-i\Phi_1} \\ t_{12}^2 e^{i\Phi_1} + t_{12}^1 e^{-i\Phi_2} & - (T^2 T^1)_{12} \end{pmatrix},$$

where $T^{1,2}$ are the transfer matrices for the two intervals forming the circle. **NB!** The TW matrix can be reconstructed up to the similarity transformation with diagonal unitary matrix

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$$= \begin{pmatrix} - (T^1 T^2)_{12} & \left( \frac{1}{t_{12}^1} + \frac{1}{t_{12}^2} e^{i\Phi} \right) e^{i(\Phi_2 + \Phi_3 - \Phi_4)} \\ \left( \frac{1}{t_{12}^1} + \frac{1}{t_{12}^2} e^{i\Phi} \right) e^{-i(\Phi_2 + \Phi_3 - \Phi_4)} & - (T^2 T^1)_{12} \end{pmatrix}.$$
No-resonance condition

No-resonance condition 1. *We say that the no-resonance condition is satisfied if and only if the D-D spectra of the SL operators on the intervals \([x_1, x_2]\) and \([x_3, x_4]\) do not intersect.*

Necessary and sufficient conditions:
- There exists an eigenfunction supported by the kernel \(\Rightarrow\) no-resonance condition is violated at this value of the energy.
- No-resonance condition is violated \(\Rightarrow\)
  - either there exists an eigenfunction supported by the kernel,
  - or the scattering matrix is diagonal (at this energy).
No-resonance condition 1. We say that the no-resonance condition is satisfied if and only if the D-D spectra of the SL operators on the intervals $[x_1, x_2]$ and $[x_3, x_4]$ do not intersect.

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- No-resonance condition is violated $\Rightarrow$
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Theorem 1. Let the no-resonance condition be satisfied. Then the potential on \( \Gamma_3 \) is uniquely determined by the TW-function \( M(\lambda, \Gamma_3) \) known for \( \Phi = 0, \pi \), where \( \Phi \) is the total flux of the magnetic field through the ring

\[ \Phi = \int_{[x_1, x_2] \cup [x_3, x_4]} a(y)dy. \]

Idea of the proof

\[
\begin{align*}
\left| (M_0(\lambda))_{12} \right| & = \left| \frac{1}{t_{12}^1} + \frac{1}{t_{12}^2} \right|, \\
\frac{1}{4} \left( \left| (M_0(\lambda))_{12} \right|^2 - \left| (M_\pi(\lambda))_{12} \right|^2 \right) & = \frac{1}{t_{12}^1} \frac{1}{t_{12}^2}.
\end{align*}
\]

\( \Rightarrow \) the analytic functions \( t_{12}^1 \) and \( t_{12}^2 \) are determined.

The entry 11 gives us the function

\[
(T^1(\lambda) T^2(\lambda))_{12} = t_{11}^1(\lambda) t_{12}^2(\lambda) + t_{12}^1(\lambda) t_{22}^2(\lambda) = -t_{12}^1(\lambda) t_{12}^2(\lambda)(M_0(\lambda))_{11}.
\]

Consider the points \( \lambda_j \) - the zeroes of \( t_{12}^1 \)

\[
t_{11}^1(\lambda_j) = (T^1(\lambda_j) T^2(\lambda_j))_{12}/t_{12}^2(\lambda_j)
\]

\( \Rightarrow \) the entire function of exponential type \( t_{11}^1 \) is uniquely determined \( \Rightarrow \) the function \( t_{22}^1 \) is determined.
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\Phi = \int_{[x_1, x_2] \cup [x_3, x_4]} a(y) \, dy.
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Consider the points \( \lambda_j \) - the zeroes of \( t_{12}^1 \)
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t_{11}^1(\lambda_j) = \frac{(T^1(\lambda_j^1) T^2(\lambda_j^1))_{12}}{t_{12}^2(\lambda_j^1)}
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$\Rightarrow$ the analytic functions $t_{12}^1$ and $t_{12}^2$ are determined.

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Consider the points $\lambda_j^1$ - the zeroes of $t_{12}^1$

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$\Rightarrow$ the entire function of exponential type $t_{11}^1$ is uniquely determined $\Rightarrow$ the function $t_{22}^1$ is determined.
Theorem 1. Let the no-resonance condition be satisfied. Then the potential on \( \Gamma_3 \) is uniquely determined by the TW-function \( M(\lambda, \Gamma_3) \) known for \( \Phi = 0, \pi \), where \( \Phi \) is the total flux of the magnetic field through the ring
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\( \Rightarrow \) the analytic functions \( t_{12}^1 \) and \( t_{12}^2 \) are determined. The entry 11 gives us the function
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(\mathcal{T}^1(\lambda) \mathcal{T}^2(\lambda))_{12} = t_{11}^1(\lambda) t_{12}^2(\lambda) + t_{12}^1(\lambda) t_{22}^2(\lambda) = -t_{12}^1(\lambda) t_{12}^2(\lambda) \langle M_0(\lambda)\rangle_{11}.
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\( \Rightarrow \) the entire function of exponential type \( t_{11}^1 \) is uniquely determined \( \Rightarrow \) the function \( t_{22}^1 \) is determined.
Theorem 2. Assume that:

- $\Gamma$ is a metric graph which is:
  - formed by a finite number of compact intervals,
  - has no loops,
  - has Euler characteristic zero, i.e. has one cycle;
- $L_{q,a}$ is the magnetic Schrödinger operator in $L_2(\Gamma)$, with
  - $q \in L_2(\Gamma)$ real,
  - $a \in C(\Gamma)$ real,
  - standard boundary conditions at the vertices;
- $\Phi$ is the total flux through the cycle;
- $M_\Phi(\lambda)$ is the TW matrix function.

Then the TW matrix function $M_\Phi(\lambda)$ known for $\Phi = 0, \pi$ determines the graph $\Gamma$ and the potential $q$, provided that the no-resonance condition is satisfied.
No-resonance condition
Let the kernel $\ker \Gamma$ be a cycle divided by contact points $\gamma_j, j = 1, 2, \ldots, \mathcal{M}$ into $\mathcal{M}$ intervals $[x_{2j-1}, x_{2j}]$.
Denote by $L_{q,a}^{j,k}|_{\ker \Gamma}, j \neq k$ the self-adjoint operator determined by the differential expression $L_{q,a}$ on the domain of functions from $\bigoplus \sum_{j=1}^{\mathcal{M}} W^2_2([x_{2j-1}, x_{2j}])$ satisfying Dirichlet boundary conditions at the contact vertices $\gamma_j$ and $\gamma_k$ and the standard boundary conditions at all other contact points.

No-resonance condition 2. *We say that the no-resonance condition is satisfied if and only if the spectrum of at least one of the self-adjoint operator $L_{q,a}^{j,k}|_{\ker \Gamma}$ is simple, i.e. no multiple eigenvalue occurs.*

Reconstruction of the potential on the branches


Then Theorem 1 implies Theorem 2.
Diolch yn Fawr!