A Combinatorical Approach to Vector Tomography for Doppler Spectral Data

K. Stråhlén
Department of Mathematics
Lund Institute of Technology
P.O. Box 118, S-221 00 LUND, Sweden

Abstract

Velocity spectra of a flow can be made by ultrasound Doppler measurements. Using only part of the information in these spectra, it is possible to reconstruct the solenoid part and the support of the flow. Here we demonstrate that using a local minimisation combinatorical algorithm and a certain neighbourhood structure it is possible to reconstruct all of the flow, hence also the divergence. In simulations the reconstructions are very close to the original full flow.

1 Spectrum theory

Consider a stationary flow inside a bounded body described by a vector valued function $f$. When a collimated ultrasonic wave of frequency $\omega_0$ and velocity $c$, $a(t) = e^{\omega_0 t}$, meets a particle of speed $v$ in the opposite direction of the wave, having the right acoustic properties, the reflected signal obtains a Doppler shift $\delta$. If $|v| \ll c$, then $\delta = kv$ with $k = \frac{2\omega_0}{c}$. The sign is reversed if the particle moves in the same direction as the wave.

If some simplifying physical assumptions are made, the reflected signal is

$$b(t,x) = \lambda e^{i(\omega_0 + kv)t},$$

where $\lambda$ is a proportionality constant. For simplicity we set $\lambda = 1$. The problems with $\lambda \neq 1$ will be discussed later. Let $L$ be a directed line with direction $\Theta \in S^{n-1}$, where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. When having a stream of particles, superposition yields

$$b(t) = \int e^{i(\omega_0 + kv)t} \sigma(f,L,v) \, dv,$$

where $\sigma(f,L,v) \, dv$ is a Radon measure, depending on the number of particles on $L$ with speed values between $v$ and $v + dv$ along $L$,

$$\sigma(f,L,v) \, dv = m^*\{x \in L | v < \Theta \cdot f(x) \leq v + dv\},$$

where $m^*$ is the Lebesgue measure on the line, or more precisely:

$$\sigma(f,L,v) \, dv = (\Theta \cdot f(x)) \, dl,$$

where $\cdot$ denotes the push forward operation.

The fundamental observation is that $\sigma$ may be expressed as the Fourier transform, $\mathcal{F}$, of $b$,

$$\sigma(f,L,\frac{\omega - \omega_0}{k}) = c \mathcal{F} b(\omega).$$

This means that the velocity spectrum $\sigma(f,L,v)$ along $L$ is obtained from a translated Fourier transform of the received signal. In Figure 1 a typical velocity spectrum is shown.

![Figure 1: A typical velocity spectrum.](image)

Consider the moments $M_n(\sigma,L) = \int_{-\infty}^{\infty} v^n \sigma(v) \, dv$. Integration over levels of $\Theta \cdot f$ yields

$$M_n(\sigma,L) = \int (\Theta \cdot f)^n \, dl.$$

The integral is a line integral along $L$. In particular,

$$M_0(\sigma,L) = \int_{-\infty}^{\infty} \chi_{\text{suppt}}(dl) = \mathcal{P} \chi_{\text{suppt}}(L),$$

$$M_1(\sigma,L) = \int_{-\infty}^{\infty} \Theta \cdot f \, dl = \mathcal{P} (\Theta \cdot f)(L),$$

where $\mathcal{P}$ is the ordinary X-ray transform, see e.g. [1]. There is a problem with $f(x) = 0$. If there are particles in $x$ with speed zero, that is $f(x) = 0$, then $x \in \text{suppf}$, a situation to be distinguished from points $y \notin \text{suppf}$, where reflecting particles are absent and where $f$ is undefined.

Reconstruction of the support from $M_0$ has been made in [2] and [3]. Reconstruction of the solenoid part of the
flow from \( M_1 \) is well known, see e.g. [4, 5, 6, 7, 3]. The divergence can not be found from \( M_0 \) and \( M_1 \).

It is not known how to reconstruct the full flow from the velocity spectra. There are even examples of flows not uniquely determined by their spectra. Let e.g. \( c \) be a central symmetric field, \( c(x) = \varphi(|x|)x \), for some smooth \( \varphi \), and \( r \) a smooth rotationally symmetric field \( r(x) = r(|x|) \).

Then \( c + r \) and \( -c + r \) will have identical spectra.

Here we present an iterative numerical algorithm for the planar case which reconstructs the full flow. In the ambiguous cases, like the ones above, the output depends on the initial flow.

2 The discrete setting

Divide the reconstruction area into rectangular pixels. Let the number of pixels be \( mn \). Let the number of lines be \( s \), where \( s \) is the number of detector positions and \( d \) the number of angles. Let \( a_i(j) \) be the length of line \( i \) within pixel \( j \). We get \( sd \) vectors \( a_i \) of length \( mn \), \( a_i = (a_i(1), \ldots, a_i(mn))^T \), \( i = 1, \ldots, sd \). Define vectors by \( b_i = (a_i^x \cos \psi_i, a_i^y \sin \psi_i)^T \), where \( \psi_i \) is the angle between line \( i \) and the positively directed \( x \)-axis, see Figure 2. \( h_i \) is a vector of length \( 2mn \), \( i = 1, \ldots, sd \). Let \( h = (h_1^T, h_2^T)^T \), where \( h_1, h_2 \) are \( mn \)-vectors representing the \( x \) and \( y \) components of the flow. \( h \) is a vector of length \( 2mn \) and is a discrete representation of the flow \( f \). Discrete versions of the spectra will be used. Let \( r \) be an even integer and

\[
S_i(k) = \int_{\frac{2\psi_a}{r}}^{\frac{2\psi_{a+1}}{r}} \sigma(f, L, v) \, dv ,
\]

see Figure 3. Thus \( S_i(k) \) is the length of the part of the line in which \( \psi_{\text{max}} \frac{2\psi_{a+1}}{r} < \Theta \cdot f(x) \leq \psi_{\text{max}} \frac{2\psi_a}{r} \).

Let

\[
\chi(j) = \begin{cases} 
    1 & \text{if } h_1(j)^2 + h_2(j)^2 > 0 \\
    0 & \text{otherwise}
\end{cases}
\]

The discrete version of the spectrum gives rise to the following discrete definitions for \( M_0 \) and \( M_1 \):

\[
M_0(L_i) = \mathcal{P}_\chi_{\text{supp}}(L_i) = a_i^T \chi = \sum_{k=-r/2}^{r/2} S_i(k)
\]

and

\[
M_1(L_i) = \mathcal{P}_\chi(\Theta \cdot f[L_i]) = b_i^T h = \sum_{k=-r/2}^{r/2} \frac{2k}{r} v_{\text{max}} S_i(k) .
\]

The problem of particles without speed discussed above is dealt with in the discrete case by changing \( h \) to a random vector of very small magnitude in pixels belonging to the support and where \( h = 0 \).

Let \( S = (S_1, S_2, \ldots, S_{sd}) \). We use the following metric, the Kantorovich metric, to measure distances between two sets of spectra \( S \) and \( \tilde{S} \):

\[
\mathcal{D}(S, \tilde{S}) = \sum_{j=1}^{sd} \sum_{k=-r/2}^{r/2} \left| (S(j) - \tilde{S}(j)) \right| .
\]

It is important that \( S_i \) and \( \tilde{S}_i \) have the same mass for each \( i \), that is

\[
\sum_{j=-r/2}^{r/2} S_i(j) = \sum_{j=-r/2}^{r/2} \tilde{S}_i(j) .
\]

In reality there is also a problem with scale. The scale tells how the length \( a_i(j) \) is related to the physical dimensions of the measurement setup, that is the actual size of the pixel. Another scale factor relates the actual value of \( S_i(k) \) to the measured quantity in volts. The scaling has e.g. the effect that

\[
M_0(L_i) = \lambda a_i^T \chi = \lambda \mathcal{P}_\chi(L, i) ,
\]

for some \( \lambda \neq 1 \). This is a problem when reconstructing \( \chi \) The scale factor can be determined from the measurement setup. However, it is also possible to determine it from the data, which will be done here.

Now we can formulate the spectral reconstruction problem in this discrete setting. Using ART-approaches, see [1], it is possible to reconstruct the support and the solenoid part of the flow.
Problem 2.1. Let $S$ be the discretisation of the known spectra and let $u$ be the solenoid field determined from $M_1$ and $M_0$. Let $s \geq 0$ be a step length. Solve the following problem

$$
\min_{g_1, g_2} \mathcal{D}(S, \hat{S}(u + (g_1, g_2))) ,
$$

where $\hat{S}(u + (g_1, g_2))$ is the discrete spectra of $\hat{h} = u + (g_1, g_2)$. Impose the constraints $\supp h \subset \Upsilon$

$g_1(j), g_2(j) \in$ $[-v_{\max}, -v_{\max}(1 - \frac{2s}{r}), \ldots, v_{\max}(1 - \frac{2s}{r}), v_{\max}]$, $j = 1, \ldots, mn$ and $|h| \leq v_{\max}$.

We define the neighbours of $h$ as the set

$N_h = \{f| \text{for one pixel } x: f(y) = h(x), y \neq x \text{ and } f_i(x) = h_i(x) + v_{\max} \frac{1}{r} \text{ or } f_i(x) = h_i(x) - v_{\max} \frac{1}{r}, i = 1 \text{ or } 2 \}$.

A neighbour of $h$ is different from $h$ only in one pixel and there it is either $v_{\max} \frac{1}{r}$ units larger or $v_{\max} \frac{1}{r}$ units smaller than $h$ in either the first or second component.

3 A minimisation algorithm

First compute the weights $a_i(j)$ and $b_i(j)$ off line once for all. They only depend on the pixel and the measurement geometries. Each step will be described further below.

(i) Determine the scale. If the scale $\lambda \neq 1$ then replace $S$ by $S/\lambda$. Use this new $S$ from now on.

(ii) Compute a start solution $h$ using $M_0$ and $M_1$. $h$ is built up from the support and solenoid part of the flow.

(iii) From now on regard the image to only consist of the pixels in the support. In this way the number of neighbours that have to be regarded in each step is reduced.

(iv) Choose step length $s$.

(v) Compute the spectra $S^m$ corresponding to $h(x)$.

1. For each element in $N_h$, update $S^m$ to $S^v$ and compute $\mathcal{D}(S, S^v)$

2. Set $S^m_{\text{old}} = S^m$. Choose the $n$ that minimises $\mathcal{D}(S, S^v)$ (if more than one, choose one) and compute the new $h$ corresponding to this $n$. Set $S^m = S^v$.

3. If $\mathcal{D}(S, S^m)$ is smaller than $\mathcal{D}(S, S^m_{\text{old}})$, go to 1.

4. If $s > \text{threshold}$, set $s = s/2$ and go to (v).

(vi) The (locally) minimising solution is $h$.

To avoid to be caught in local minimas the above algorithm can be adjusted to a simulated annealing approach. However, for the tested flows, our algorithm has worked without this modification.

3.0.1 Determining the scale

The scale can be determined from the measurement data. First reconstruct the support from $M_0$. This is an operation independent of scale since we threshold the image to find function that is either 1 or 0. In reality due to the scale problems it should be either $\lambda$ or 0, see (1). Thus by comparing $M_0(L_i)$ with the X-ray transform of the reconstructed support $\hat{S}Y$ for all $L_i$ it is possible to determine the scale.

It is also possible to judge the quality of the reconstructed support, by checking the variance of this scale as a function of beam number.

3.0.2 Computation of the start solution

The start solution is computed using a modified ART algorithm, which uses the information of $M_0$ and $M_1$ to reconstruct the vorticity of the flow. First the support is reconstructed using ART. The image is thresholded to find a subset of the $mn$ pixels containing the support. Another reconstruction only using these pixels is made with ART and the threshold that gives the support best fitting the data is chosen. Thereafter the solenoid part is reconstructed using only the pixels in the support thus found. For some simulation demonstrating this algorithm see [8].

3.0.3 Updating the spectra

Having $S^m$ corresponding to $h$, it is easy to determine $S^m$ corresponding to a neighbour $\hat{h}$ of $h$, as $h$ and $\hat{h}$ differ only in one pixel $x$. Determine all lines going through the pixel $x$. For each such line, $L_{ij}$, determine the quantity $z = h(x) \cdot \hat{e}_{ij}$ and $\hat{z} = \hat{h}(x) \cdot \hat{e}_{ij}$. To find $S^m_{ij}$, remove $a_{ij}(x)$,
from the position in \( S_{m}^{n} \) in which \( z \) lies and add it to the position in \( S_{m}^{n} \) in which \( \bar{z} \) lies, see Figure 4.

### 3.0.4 Updating of the Kantorovich distance

The distance \( D(S, S^{n}) \) is stored as a vector
\[
(D(S, S^{(1)}), \ldots, D(S, S^{(n)}(L_{j}))).
\]

Only the distances corresponding to the lines \( L_{j} \) have to be recomputed.

### 3.1 Erroneous support

If the support is not correctly reconstructed the mass of the spectrum of the start solution and the measurements for corresponding lines is not the same, which is a condition when using the Kantorovich metric. Our approach to beat this problem is to add the difference in mass between two corresponding spectra to the position corresponding to speed 0 to one one of them, so that the masses become equal. Thus pixels reconstructed to be in the support, which in fact outside it, is assigned a 0-vector. This does not change the appearance of the flow. Pixels reconstructed to be outside the support, which is in fact in it, are (erroneously) regard as having speed 0. There is no hope of reconstruction in these pixels anyway. This approach is used with good results here.

### 4 Simulations

Consider a 20 \times 20 pixel flow. The spectra were computed for 45 \times 45 lines in accordance with the fan-beam geometry. We used \( s = 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, r = 20 \).

The support was correctly reconstructed by the algorithm. For the initial solution the cost \( D \) is approximately 1709. For the final solution it is approximately 120. The error measure \( M = \sum_{i=1}^{20} \sum_{j=1}^{20} ((X(i, j) - \bar{X}(i, j))^{2} + (Y(i, j) - \bar{Y}(i, j))^{2})^{1/2} \),

with \((X, Y)\) the correct flow and \((\bar{X}, \bar{Y})\) the reconstructed, decreases from approximately 8.97 for the initial guess to 1.30 for the final reconstruction. For a standard ART reconstruction using only \( M_{1} \) it was 44. See Figures 5-6. Note that the residuals after having reconstructed the solenoid part describe the divergence very well. The residual values have been enlarged with a factor five for clarity.

The algorithm was tested for a number of start solutions, which differed by potential flows. It found the correct flow in these cases, see Figure 7-8. It was also tested for the ambiguous case. The field \(|\phi(k)| \ast \phi(|k|)\), with \( \phi \) a smoothed version of the characteristic function of an interval of length 12, was used. In this case it found flows highly dependent of the start solution, with random appearances, see Figure 9.
4.1 Erroneous support

In this example, the support has been reconstructed incorrectly. The simulation was done on a 20 × 20 flow, with 45 × 45 measurement beams. $v_{\text{max}} = 1.2$ and $r = 20$ were used. Here $\mathcal{D}$ falls from 5966 to 4700, which is not as dramatic as when the support was correct. However, the measure $\mathcal{M}$ decreases from 9.96 to 2.60 during the algorithm. For the standard ART method using only $M_1$ it is 44.8. As a comparison the distance $\mathcal{D}$ from the correct flow to the flow we receive by restricting the correct flow to the reconstructed support and adding a very small flow to the pixels erroneously reconstructed to be inside the support is 4628. This is the subjectively ‘best’ possible solution. Thus, the algorithm still improves the reconstruction significantly and comes very close to the minima in spite of the erroneously reconstructed support. See Figures 10-12. The residuals have been enlarged with a factor five.

![Figure 10: Left: Original flow. Right: 5xResidual with ordinary ART.](image1)

![Figure 11: Left: Support of correct flow. Right: Reconstructed support.](image2)

![Figure 12: Left: 5xResiduals for initial guess. Right: 5xResiduals for final reconstruction.](image3)

5 Conclusions

In this paper we have developed an algorithm, which determines the full flow field, using the velocity information from Doppler tomography spectral measurements. Simulations suggests that this algorithm can be used to obtain good reconstructions of the full flow, which has not earlier been possible.

Acknowledgements

I thank Kalle Åström for helping me with the combinatorical optimisation needed in this paper. I also thank Rikard Berthilsson for sharing his knowledge in measure theory.

References


