Fractal Structures in Dyadic Diophantine Approximation

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Abstract

The thesis takes as starting point coverings of the unit circle $\mathbb{S}$ driven by a dynamical system, how this is connected to diophantine approximation and the study of continued fractions. The special case of letting the system follow the diadic numbers will be discussed. We prove that the set $\{x \in \mathbb{S} : 2^n x \geq c, n > 0\}$ is a fractal set whose Hausdorff dimension depends continuously on c and is constant on intervals which form a set of Lebesgue measure 1. Hence it has a fractal graph. We give a way of completely characterise the intervals where the dimension is constant.

We continue by studying the set $\{x \in \mathbb{S} : 1 - c \geq 2^n x \geq c, n > 0\}$ and prove that this shows similar properties to the set bounded from one side only. We prove that in this two-sided case we have connections to the field of combinatorics on words and that the dimension is zero if and only if $c \geq 1 - 2\tau$, where τ is the Thue-Morse constant.

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Contents

1. Introduction

The thesis is devoted to the study of the classical problem of circle covering, and how this, in our special case, gives a link to the areas of diophantine approximation and to combinatorics on words. These are areas which have been deeply and intensively studied, and we shall shortly give a brief presentation of some classical results and landmarks achieved therein.

We present the study of two different covering-models. The first one, we completely characterises in the sense of computability, while the second leaves us with some unanswered questions.

The thesis is organised as follows; we continue this chapter by giving an introductive presentation of some classical results in the area of circle covering. Thereafter the connection of our work to the area of diophantine approximation is discussed. In the two sections thereafter we introduce two covering models, the one-sided and the two-sided, we have studied and present the results achieved. In the second chapter we give necessarily definitions for our work and present some classical theorems and introduce the notation we are going to use. We end the chapter by giving a short presentation of the β-shift. Chapter three is devoted to the study of minimal sequences, a useful tool for our forthcoming work, their properties and state how to characterise them. The fourth chapter is like the second chapter spent on giving necessary material for the coming work. We introductively present here the area of symbolic dynamics. In chapter five we turn to presenting our main results and we end the study of the first covering-model by giving some numerical aspects in chapter six.

Thereafter, in chapter seven, we turn our attention to the second covering-model. At the very end we briefly discuss some questions for extending the work.

1.1 Covering the circle

Let us by $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ denote the unit circle, which we may identify with the unit interval $[0, 1)$, and let $\{I_n\}$ be a sequence of open intervals, not necessarily disjoint, on $\mathbb{S}$. The length of an interval I_n will be denoted by l_n. We attach the intervals to a stochastic process $\{X_n\}$, taking values in $\mathbb{S}$, controlling the distribution of them.

The classical problem, known as, *circle covering problem* is a problem originating in the mathematical field of geometric probability. The question is to see under which conditions the circle $\mathbb{S}$ will be covered, partly or completely, when distributing the intervals $\{I_n\}$ along the circumference.

A version of the circle covering problem is the special case when the intervals are only finitely many and they all have the same length. This problem is known in literature as the *bicycle wheel problem*, (see e.g. [6]). A more precise formulation and a motivation of the name of the problem is the following:

> A man is cycling along a road and passes through a region strewn with tacks; he wishes to know whether one has entered his tyre. Because of the traffic, he can only snatch glances at random times. At each glance he covers a fraction a of the wheel. What is the probability that after n glances he has seen the whole wheel? In mathematical terminology: n intervals are placed randomly on a circle, each covering a fraction $a = l_j$, $j = 1 \ldots n$, of the circle. What is the probability that the circle is completely covered?

In a paper from 1939 Stevens [23] presents, using a combinatorial argument, the solution to the problem as

$$P(a, n) = \sum_{k=0}^{\lfloor 1/a \rfloor} (-1)^k \binom{n}{k} (1 - ak)^{n-1},$$

where $\lfloor 1/a \rfloor$ is the integral part of $1/a$. Even-though the result was presented by Stevens it was partially obtained before by

Whitworth, Baticle and others, for more of this see [6]. Stevens also derived, a solution to question of finding the probability of having precisely i gaps in the covering.

In the more general case, when the intervals are infinitely many and may have different length, we make the assumption that $\{X_n\}$ is an independent identically distributed (i.i.d) and uniform process. Under this prerequisites the Borel-Cantelli lemma implies that almost every point, (in the sense of Lebesgue measure), is covered infinitely often with probability one if and only if we have

$$\sum_{n=1}^{\infty} l_n = \infty. \tag{1.1}$$

If, on the other hand, the sum in (1.1) is convergent the lemma implies that almost every point on $\mathbb{S}$ is covered finitely many times with probability one.

Taking the step from considering the question of when a.e. point is covered infinitely often into the question of when all points are covered infinitely often with probability 1 one enters the problem, which is known as the *Dvoretzky covering problem*. Mathematically formulated the question is to formulate conditions of the interval to have

$$P\Big(\mathbb{S} = \bigcup_{n=1}^{\infty} I_n\Big) = 1.$$

This problem was raised in 1956 by Dvoretzky [7] who noticed that if the sum in (1.1) does not diverge too fast there are, with positive probability, points on $\mathbb{S}$ which will not fall into any interval I_n, and on the other hand, if the divergence is not too slow this does not occur. Dvoretzky also gave the first example of a sequence of intervals where $\sum_{n=1}^{\infty} l_n$ diverges but the intervals do not cover the whole circle with probability one.

The Dvoretzky covering problem was further considered by

Billard [4] and Kahane [11] who found

$$\limsup_{n\to\infty} \frac{1}{n} \exp(l_1 + l_2 + \ldots + l_n) = \infty \tag{1.2}$$

to be a sufficient condition on the intervals for covering, and

$$\sum_{n=1}^{\infty} l_n^2 \exp(l_1 + l_2 + \ldots + l_n) = \infty \tag{1.3}$$

beeing a necessary one.

A special case of the interval drawing attention is when having their lengths being of the form $l_n = \frac{c}{n}$. By the condition (1.2) it follows that when $c > 1$ the whole unit circle is infinitely covered, while (1.3) implies that this is not the case for $0 < c < 1$. Billard's and Kahane's results leave the case $c = 1$ unsettled. Furthermore Kahane proved that under this choice of l_n the set F_c of finitely covered points is of Hausdorff dimension[1],

$$\dim_H F_c = 1 - c \tag{1.4}$$

for $0 < c < 1$.

The general covering question was further considered by Orey (unpublished) who, using topological as well as probabilistic methods to study the number of components of the union of the first n arcs, improved (1.2) to the stronger condition

$$\limsup_{n\to\infty} \frac{1}{n} \exp(l_1 + l_2 + \ldots + l_n) > 0.$$

Independently of Orey's work Mandelbrot [15, 16] deduced a similar result. Orey's and Mandelbrot's improvement showed that, in the special case discussed above, $c = 1$ also gives an infinite covering.

[1] See definition on page 13.

In 1973 Shepp [22] closed the question of finding restrictions of the intervals when proving the condition, for a sequence $\{l_n\} \searrow 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(l_1 + l_2 + \ldots + l_n) = \infty$$

to be both a necessary and sufficient one for covering all points on $\mathbb{S}$ with probability 1.

This was however not the very end of the Dvoertzky covering problem nor the original circle covering problem. Different version of the question and modifications of the prerequisites have attracted the attention of many. Among others Carleson, Kahane and Fan (see [9]) posed the question of how many times a point is covered. In [9] Fan examined the set A_β, defined to be the set of points covered by a number $\beta \log n$ times of the first n randomly placed intervals. More precise the set A_β is defined as

$$A_\beta = \left\{ x \in \mathbb{S} : \lim_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} \chi_{I_j}(x) = \beta \right\},$$

where χ is the characteristic function. By letting the length of the intervals be $l_n = c/n$, $(c > 0)$ Fan proved the existence of certain intervals of $\beta > 0$ where the Hausdorff dimension of the set A_β is equal to

$$\dim_H A_\beta = 1 - (\beta \log(\beta/c) - (\beta - c)),$$

which implies that points on the circle are differently covered.

Fan and Schmeling [10] dropped the condition of having the intervals $\{I_n\}$ attached to the an i.i.d. stochastic process X_n. They considered the case of having the covering process driven by an ergodic dynamical system. The special system used were a rotation of an irrational, that is having the intervals centred at

$$X_n = x + n\alpha \pmod 1$$

with $\alpha \notin \mathbb{Q}$ and with the length of the intervals

$$l_n = l_n(c, \mu) = \frac{c}{n^\mu}$$

where $c > 0$ and $\mu > 0$. When having $\mu = 1$, they prove that the set F_c of finitely covered points has Hausdorff dimension 0 for $c > \frac{1}{2}$, which shows a significant difference to the randomly distributed case, compare to (1.4).

Like in the work of Fan and Schmeling we are going to consider covering models where $\{X_n\}$ is no longer an i.i.d. process. Our models will be driven by the dyadic numbers, numbers of the form $\frac{k}{2^m}$ and where, the interval lengths are of the form $\frac{c}{2^m}$. We are interested in the set of finitely covered points and we will prove that the dimension of this set behaves in a continuous but fractal manner.

A rigourous explanation of the models and presentation of the results achieved follows in the coming subsections.

1.2 Application to diophantine approximation

Let (X, d) be a metric space. Given a sequence $\{x_n\} \subset X$, of maybe random numbers, and a sequence $\{l_n\}$ of positive real numbers we define the following two sets

$$I = \{\, y \in X : d(x_n, y) < l_n, \text{ infinitely many } n \,\},$$

and

$$F = X \setminus I.$$

By the notion diophantine approximation we shall mean the study of the sets I and F. Let us make the following remark: if the sequence $\{x_n\}$ is dense in X then I is a non-empty and residual set in the sense of Baire.

Example 1.1 Let us define the sequence,

$$\{x_{n,m}\}_{n \in \mathbb{N},\, 0 \leq m < n} \quad \text{with} \quad x_{n,m} = \frac{m}{n} \tag{1.5}$$

and with the particular choice of the sequence $\{l_n\}$,

$$l_n = \frac{1}{n^\alpha}. \tag{1.6}$$

For this special choices of $\{x_n\}$ and $\{l_n\}$ we are in the case of the classical diophantine approximation with rational numbers. It is a well know fact that if $\alpha > 2$ then F is non-empty. □

Inspired by the above example, we continue in this direction and refine the definition of the set F to be the following set

$$F(\alpha) = \left\{ y \in X : d(x_{n,m}, y) < \frac{1}{n^\alpha}, \text{ finitely many } n \right\}. \tag{1.7}$$

An interesting question is to look at the critical exponent, α_0, such that $F(\alpha)$ is empty if $\alpha < \alpha_0$ and is non-empty when $\alpha > \alpha_0$. For this special value α_0 we say that the set $F(\alpha_0)$ is the set of *Badly approximable numbers*, *BAN*.

A second step in refinement of (1.7) is to introduce the dependence on an extra parameter, c,

$$F_c(\alpha) = \left\{ y \in X : d(x_{n,m}, y) < \frac{c}{n^\alpha}, \text{ finitely many } n \right\}.$$

In the one-dimensional case this refinement leads to the area of continued fraction, which first systematically studied by the Dutch astronomer Huygens in the 17-th century, motivated by technical problems while constructing a model of our solar system. Briefly, the continued fraction for a real x is,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where a_n are called *partial denominators*. For brevity the continued fractions is often denoted by $[a_0, a_1, a_2, \dots]$. The following theorem gives a neat connection between the badly approximable numbers and the continued fractions, for a proof see [13].

Theorem 1.2 *An irrational x is a BAN if and only if its partial denominators are bounded.*

Yet another version, or refinement, of the F set can be introduced via a condition on the partial denominators. We set

$$F_N(2) = \{ x : x = [a_0, a_1, a_2, \ldots] \text{ with } a_j < N \}.$$

The theory of iterated function system, *IFS*-theory, gives an implicit formula for the $\dim_H F_N(2)$. The set $F_c(2)$ is finer as $F_N(2)$ counts only the maximal a_i while the c takes into account all a_i. In 1891 Hurwitz found that if $c < \frac{1}{\sqrt{5}}$ then $F_c(2)$ is empty and moreover the constant c is the best possible, but otherwise little is known about the set $F_c(2)$.

In this thesis we are going to study a special case of diophantine approximation, approximation by dyadic rationals. Similar to (1.5) and (1.6) we set the sequences $\{x_n\}$ and $\{l_n\}$ to be

$$x_{n,m} = \frac{m}{2^n}, \quad \text{and} \quad l_n = \frac{c}{2^n}.$$

We will turn our interest to the same type of questions as in the more general and classical approximation case, and look at the set of badly approximable numbers under the dyadic case.

1.3 A one-sided dyadic covering model

For an integer $n \geq 1$ there are unique integers $m \geq 0$ and $k \geq 1$ such that we may write n in the form, $n = 2^m + k$ with $0 \leq k < 2^m$. Using this we define

$$X_n = \frac{1}{2^{m+1}} + \frac{k}{2^m} = \frac{1 + 2k}{2^{m+1}}. \tag{1.8}$$

We choose to consider the case with only odd nominator, as a dyadic number with even nominator falls into a number with a smaller denominator. For some positive constant $0 < c < 1$ we put

$$l_n = l_n(c) = \frac{c}{2^{m+1}}. \tag{1.9}$$

and we let the intervals $\{I_n\}$ be of the form

$$I_n = I_n(l_n) = X_n + (0, l_n). \tag{1.10}$$

In this covering a point that will fall into an interval must fulfil the inequalities

$$0 < x - X_n < l_n$$

for some n. Writing out the defined notation gives

$$0 < x - \frac{1}{2^{m+1}} - \frac{k}{2^m} < \frac{c}{2^{m+1}}.$$

Hence a point x is covered if

$$0 < 2^{m+1}x - 1 - 2k < c \quad \text{for some } n = 2^m + k \tag{1.11}$$

is fulfilled. As c is between 0 and 1 we see that the points x which are covered, i.e. which fulfils the above condition, they also fulfil the condition

$$2^q x \ < \ c \pmod 1, \quad \text{for some } q \geq 1. \tag{1.12}$$

Conversely, a point fulfilling (1.12) also fulfils (1.11). To see this, let x be such that (1.12) holds. Then there is an integer p, which we write in the form $p = 2^j(1 + 2k)$, that is

$$0 < 2^q x - p < c$$

This implies

$$0 < 2^q x - p = 2^q x - 2^j(1 + 2k) = 2^j\left(2^{q-j}x - 1 - 2k\right) < c,$$

hence (1.11) holds with $m = q - j - 1$, and the equivalence between (1.11) and (1.12) follows.

We are interested in those points of the unit circle that never will fall into an interval, i.e. those points which never are covered. In the regard of (1.12) we have a clear condition for characterising them, and we define the set $\mathcal{S}_c$ accordingly,

$$\mathcal{S}_c = \{\, x \in \mathbb{S} \;:\; 2^n x \geq c \pmod 1, \text{ for all } n \geq 1 \,\}. \tag{1.13}$$

The main result in this work is the characterisation of the set $\mathcal{S}_c$. We introduce the map

$$\phi : (0,1) \to [0,1], \quad \text{such that} \quad \phi(c) = \dim_H \mathcal{S}_c$$

where $\dim_H$ denotes the Hausdorff dimension. It is clear from (1.13) that ϕ is monotonically increasing.

Previous results in this area are to be found in a paper by Urbanski [24]. Therein Urbanski presents some dimensional properties of the more general set,

$$K_g(c) = \{\, x \in \mathbb{S} \;:\; g^n(x) \geq c \text{ for all } n \geq 1 \,\}$$

where $g : \mathbb{S} \to \mathbb{S}$ is a C^2 expanding and orientation preserving map. The results include the continuity of the map $c \mapsto \dim_H K_g(c)$, and that this map is constant almost everywhere with respect to the Lebesgue measure, but leaves out the complete characterisation. We reprove these result in a simpler and elementary way.

The first part of the thesis is devoted to the study and the complete characterisation of the map ϕ, and reproving Urbanski's theorems of the Cantor-like behaviour in our special case. As will be shown, this characterisation involves certain intervals $\mathcal{I}_c$ on which ϕ is constant, (see Definition 3.3 at page 20 for the definition). Opera prima on the one-sided covering are the following:

- The map ϕ is constant a.e, with respect to the Lebesgue measure, Theorem 3.13.

- The complementary zero-set, to where ϕ is constant, has full Hausdorff-dimension, Theorem 3.14.
- The complete characterisation of minimal sequences via the left-shift, Theorem 3.21.
- The independence of the choice of a representative when characterising the intervals $\mathcal{I}_c$, Theorem 5.2.
- The maximality of $\mathcal{I}_c$, i.e. the constructed intervals are the largest intervals with constant dimension, Theorem 5.6.
- The map ϕ is continuous, Theorem 5.8.
- The computability of the map ϕ.

The graph of the map ϕ describes a Cantor staircase which is an interesting comparison to the linearity in Kahane's result (1.4), see Figure 1 at page 56 for a visualisation of the map.

1.4 A two-sided dyadic covering model

We present here our second model to be considered. It is simply a slight perturbation of the one-sided model.

We reuse the most of the definition from the previous section, that is we let the distribution of the reference points X_n be given by (1.8) and let the sequence $\{l_n\}$ of real numbers again be the one given by (1.9), for some constant $0 < c < 1$. But now we do the slight change, compared to (1.10), and put the intervals $\{I_n\}$ be of the form

$$I_n = X_n + \frac{1}{2}(-l_n, l_n).$$

A point $x \in \mathbb{S}$ is now covered by this two-sided model if for some positive integer n the inequality

$$|x - X_n| < l_n$$

is fulfilled. This condition can in the same way as in the one-sided case be reformulated into saying that a point x is covered if and only if

$$|2^q x| < \frac{c}{2} \pmod 1, \quad \text{for some } q \in \mathbb{N}.$$

holds. It follows that we can define the set of never covered points as the set

$$\mathcal{S}_c^* = \{ x \in \mathbb{S} \,:\, |2^n x| \geq \frac{c}{2} \pmod 1, \text{ for all } n \geq 1 \}. \tag{1.14}$$

Notice that $\mathcal{S}_c^*$ is the empty set if $c > \frac{1}{2}$. From the above definition it is clear that we may write $\mathcal{S}_c^*$ as the intersection

$$\mathcal{S}_c^* = \mathcal{S}_{c/2} \cap (1 - \mathcal{S}_{c/2}). \tag{1.15}$$

which turns out to be a central connection between the one-sided and the two-sided models. Like in the one-sided case we define the map $\phi^* : (0,1) \to [0,1]$ such that

$$\phi^*(c) = \dim_H \mathcal{S}_c^*.$$

The question now arising is whether the change of attachment of $\{I_n\}$ to $\{X_n\}$ disturbed the distribution of the intervals in such a way that the dimension of the sets of never covered points will be affected. More simply that is, does the equality

$$\dim_H \mathcal{S}_c = \dim_H \mathcal{S}_c^* \tag{1.16}$$

hold? We shall see that the answer is no in the general case, Theorem 7.3, but yes for some special values of c, Theorem 7.7. We shall also see that the change of the suspension will drastically change the shift.

The second part of the thesis is devoted to the the study of the map ϕ^* and enlightens some of its characteristics, which turn out to show some similarities to the behaviour of ϕ.

The section is closed by a tour into the area of combinatorics on words and substitutions leading to our main results concerning the two-sided model, namely

- The value $\phi^*(c)$ is zero if and only if $c \geq 2(1-2\tau)$, where τ is the Thue-Morse constant, Theorem 7.16.

As in the one-side case we present a picture of the graph of ϕ^*, see page 64.

2. Prerequisites

The main idea in deducing our results is to transform the problem into a problem concerning sequences. We start by giving some basic notation and to clarify the concepts in the area of dealing with sequences that will be used. Secondly we give a brief introduction to β-shifts.

2.1 Basic notation

Let us start with the notion of Hausdorff dimension,

Definition 2.1 *Let $s \in [0, \infty]$. The s-dimensional Hausdorff measure $\mathcal{H}^s(Y)$ of a subset of a metric space X is defined by*

$$\mathcal{H}^s(Y) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{1=1}^{\infty} \operatorname{diam}(U_i)^s : Y \subset \bigcup_{i=1}^{\infty} U_i \, , \, \sup_i \operatorname{diam}(U_i) \leq \varepsilon \right\}.$$

The unique s_0 such that

$$\mathcal{H}^s(Y) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0 \end{cases}$$

we call the Hausdorff dimension of the Y and it will be denoted by $\dim_H Y$.

A way of estimating the Hausdorff dimension of a set is to use the connection between the Hölder exponent and the Hausdorff dimension. The following result is well known.

Proposition 2.2 *Let $X \subset \mathbb{R}^n$ and suppose that $f : X \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha} \quad (x, y \in X).$$

Then $\dim_H f(X) \leq \frac{1}{\alpha} \dim_H X$.

For a deeper discussion of dimension theory and methods used therein see Falconer's book [8].

Expressing a real number $x \in \mathbb{S}$ into base 2 gives that it can be written as the infinite sum

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \qquad \text{with } x_i \in \{0, 1\}. \tag{2.1}$$

We see that we may identify a real number $x \in [0, 1]$ with an element $\underline{x}$ of $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$, that is, $x \leftrightarrow \underline{x}$. This coding is one-to-one except for a countable set where it is two-to-one. Since we are going to consider Hausdorff dimension we can neglect this ambiguity and work in the coding space Σ_2 equipped with the product topology. We define the metric

$$\delta(\underline{u}, \underline{v}) \;=\; \inf\left\{\frac{1}{2^k} : u_0 = v_0, \ldots, u_k = v_k\right\}$$

The word *sequences* will by default refer to binary sequences, if not explicitly stated otherwise. We shall make use of the name sequences both for elements in Σ_2 and for elements in $\{0, 1\}^N$, $N \in \mathbb{N}$, that is, sequences of finite length. For a finite sequence $\underline{s}$ we use the notation $|\underline{s}|$ to denote the length of $\underline{s}$. For a finite set the same notation will denote its cardinality.

By $\underline{s}(a, b)$ we shall mean the subsequence starting at position a and ending at b, that is $\underline{s}(a, b)$ is a sequence of length $b - a + 1$, or by shorter notation $|\underline{s}(a, b)| = b - a + 1$. If $a = 1$ we omit the parameter and simply write

$$\underline{s}(1, b) = \underline{s}(b) = s_1 s_2 s_3 \ldots s_b.$$

A *prefix* of $\underline{s}$ is a sub-sequence starting at the first position of $\underline{s}$. Similarly, for a finite sequence $\underline{s} = \underline{s}(n)$, a *suffix* is a sub-sequence that ends $\underline{s}(n)$.

Example 2.3 To illustrate the concepts of sub-sequences, prefix and suffix, consider the following sequences $\underline{s}$ of length 36,

$$\underline{s} = \underline{s}(1,36) = \underline{s}(36) = \underbrace{001101}_{\text{a prefix}}00101\underbrace{100010}_{\underline{s}(12,17)}0101110\underbrace{100100111001}_{\text{a suffix}}$$

For the sub-sequence $\underline{s}(12,17) = 100010$ we have that the length to be $|\underline{s}(12,17)| = 17 - 12 + 1 = 6$. □

For two sequences $\underline{s}$ and $\underline{u}$ the concatenation is the sequences $\underline{su}$, that is $\underline{su} = s_1 s_2 \ldots s_n u_1 u_2 \ldots$, where it was assumed that $\underline{s}$ was of finite length. The notation $\underline{s}^k$ will refer to self-concatenation k-times of a finite sequence, that is $\underline{s}^k = \underline{ss} \ldots \underline{s}$. Extending this gives that the infinite concatenation $\underline{s}^\infty$ of a finite sequences is an element in Σ_2.

Example 2.4 Let us denote two sequences by $\underline{s} = 01$ and $\underline{u} = 001$. Then two concatenations will be $\underline{su} = 01001$ and $\underline{us} = 00101$. Similarly $\underline{s}^2 = \underline{ss} = 0101 = (01)^2$. This can be generalised to

$$\underline{s}^\infty = (01)^\infty = 01010101\ldots$$

which is an element of Σ_2. □

The ordering of sequences will be the standard lexicographic order, which is, $\underline{s} > \underline{u}$ if and only if there is a 1 in $\underline{s}$ at the first position where they differ. If the sequences are of different length, we simply append a suitable amount of zeros at the end of the shorter. Note that this will imply that the sequences 0 and 00 are the same. We disregard from this type of collisions as they will cause us no trouble.

Example 2.5 By our definition of ordering we have $001 < 010$, $001 = 00100$ and $010 > 00100$. Note in particular the special case when the shorter sequences is a prefix of the longer one $001 < 00101$. $\square$

For a sequence $\underline{s}$ of length n we by the notation $[s_1 s_2 \dots s_n]$ shall mean a *cylinder set*, that is, the set such that

$$[\,\underline{s}\,] = [s_1 s_2 \dots s_n] \;=\; \{\, \underline{x} \in \Sigma_2 :\; x_i = s_i\,, i = 1, \dots, n\,\}.$$

We denote by $\sigma : \Sigma_2 \to \Sigma_2$ the left-shift of a sequence, that is,

$$\sigma(\,\underline{s}\,) = \sigma(s_1 s_2 s_3 \dots) = s_2 s_3 s_4 \dots$$

and by $\sigma^n(\,\underline{s}\,) = \sigma^{n-1}(\sigma(\,\underline{s}\,))$ the n-th recursive composition. It is quite immediate form (2.1) that multiplication by 2 on the unit circle corresponds to left-shift on the set of sequences Σ_2,

$$2x = \sum_{i=1}^{\infty} \frac{x_i}{2^{i-1}} = \sum_{i=1}^{\infty} \frac{x_{i+1}}{2^i} \pmod 1.$$

Recalling the definition of the set $\mathcal{S}_c$ in (1.13) we can therefore define its corresponding sequence version as the set

$$\mathcal{S}_{\underline{c}} = \{\, \underline{x} \in \Sigma_2 :\; \sigma^n(\,\underline{x}\,) \geq \underline{c}\,,\ \text{for all } n \geq 1\,\}. \tag{2.2}$$

From the above set we define the related set of prefixes, that is for a sequence $\underline{c}$ let $\mathcal{S}_{\underline{c}}(n)$ be the set of prefixes of length n of sequences in $\mathcal{S}_{\underline{c}}$,

$$\mathcal{S}_{\underline{c}}(n) \;=\; \left\{\, \underline{x}(n) : \underline{x} \in \mathcal{S}_{\underline{c}} \,\right\}.$$

Definition 2.6 *We define the topological entropy h_{top} of a set $\mathcal{S}_{\underline{c}}$ as the exponential growth rate of the number of sequences allowed as the length n increases,*

$$h_{\text{top}}(\mathcal{S}_{\underline{c}}) \;=\; \lim_{n\to\infty} \frac{1}{n} \log \left|\mathcal{S}_{\underline{c}}(n)\right|.$$

The existence of the above limit follows by simply noticing the sub-additivity property of the function $n \mapsto \log \left|\mathcal{S}_{\underline{c}}(n)\right|$:

$$\log \left|\mathcal{S}_{\underline{c}}(n+m)\right| \;\leq\; \log \left|\mathcal{S}_{\underline{c}}(n)\right| + \log \left|\mathcal{S}_{\underline{c}}(m)\right|.$$

There is a neat connection in our case between the Hausdorff dimension and the topological entropy,

$$\dim_H \mathcal{S}_{\underline{c}} = \frac{1}{\log 2}\, h_{\mathrm{top}}(\mathcal{S}_{\underline{c}}).$$

We will return to this connection in Theorem 5.4.

We end the section by introducing a strong property in asymptotic behaviour related to the regularity of recurrences, the notion of mixing.

Definition 2.7 *A topological dynamical system $f : X \to X$ is called topological mixing if for any two open nonempty sets $U, V \subset X$ there exists an N such that for any $n > N$ the intersection $f^n(U) \cap V$ is nonempty.*

2.2 The β-shift

The field of β-shift originates in the late fifties by Rényi [20] who introduced the representation of a real number with an arbitrary base $\beta > 1$. One of the most studied problems in this field is the link between expansions to base β and ergodic properties of the corresponding β-shift.

More precisely, the definition of the β-expansion, where $[\cdot]$ means the integral part, is the following;

Definition 2.8 *The expansion of a number $x \in [0,1]$, or β-expansion, in base β is a sequences $\underline{x}$ of integers out of $\{1, 2, \ldots, [\beta]\}$ such that*

$$x_n = [\beta T_\beta^{n-1}(x)], \qquad n \geq 1,$$

where $T_\beta : [0,1) \to [0,1)$ is the transformation $T_\beta(x) = \beta x \;(mod\; 1)$.

Definition 2.9 *The closure of the set of all β-expansions of $x \in [0,1)$ is called the β-shift, S_β.*

The expansion of one in base β, we denote it by 1_β, turns out to be crucial for characterising the β-shift. Parry [17] proved that S_β is totally determined by the expansion of 1.

Theorem 2.10 (Parry) *If 1_β is not finite (i.e. it will not terminate with zeros only), then $\underline{s} \in \{1, 2, \ldots, [\beta]\}^{\mathbb{N}}$ belongs to S_β if and only if*

$$\sigma^n(\,\underline{s}\,) < 1_\beta\,, \qquad \text{for all } n \geq 1.$$

If $1_\beta = \underline{i} = i_1 i_2 \ldots i_M 0^\infty$ then $\underline{s}$ belongs to S_β if and only if

$$\sigma^n(\,\underline{s}\,) < (i_1 i_2 \ldots i_{M-1}(i_M - 1))^\infty\,, \qquad \text{for all } n \geq 1.$$

Moreover Parry proved the following theorem.

Theorem 2.11 (Parry) *A sequence $\underline{s}$ is an expansion of 1 for some β if and only if*

$$\sigma^n(\,\underline{s}\,) < \underline{s}\,, \qquad \text{for all } n \geq 1,$$

and then β is unique. Moreover the map $\Xi : \beta \mapsto 1_\beta$ is monotone increasing.

Let $1 < \beta_1 < \beta_2$. Then all sequences of expansions of one, 1_β, for $\beta_1 < \beta < \beta_2$ are at the same time expansions for some $x \in [0,1)$ in base β_2. Let us denote this set of those x by $I(\beta_1, \beta_2)$. If $\pi : [0,1) \to S_{\beta_2}$ is the map assigning to each $x \in [0,1)$ its β_2-expansion we define the map, (depending on β_1 and β_2),

$$\rho : I(\beta_1, \beta_2) \to [\beta_1, \beta_2] \tag{2.3}$$

by putting $\rho(x)$ to be the unique β having $\pi(x)$ as it expansion of 1, that is $\pi(x) = 1_\beta$. In [21] Schmeling calculated the Hölder-exponent for the map ρ;

Theorem 2.12 (Schmeling) *The map (2.3) satisfies the Hölder condition*

$$|\rho(\underline{u}) - \rho(\underline{v})| \;\leq\; C\,\delta(\underline{u}, \underline{v})^{\ln\beta_1/\ln\beta_2}.$$

3. Minimal Sequences

This section is devoted to the study of special properties of sequences and sets thereof. We will closely investigate the set $\mathcal{S}_{\underline{c}}$ and its dependence on the structure of the sequence $\underline{c}$. As will be seen, this characterisation will heavily depend on the prefixes of $\underline{c}$ and its properties. The main effort will be made on developing tools used for later sections. We start with an easy example.

Example 3.1 For some special sequences $\underline{c}$ the set $\mathcal{S}_{\underline{c}}$ is quite easy to describe. The two trivial examples are when $\underline{c}$ is 0 or 1. The first one gives the whole set Σ_2 while the second one gives the singleton set $\{1^\infty\}$. If letting $\underline{c_k} = 0^k 1$ we have

$$\mathcal{S}_{\underline{c_k}} = \{\, \underline{x} \in \Sigma_2 : \underline{x} \text{ does not contain } k+1 \text{ consecutive zeros}\}.$$

This follows by considering what will happen when a block of zeros is shifted up to the front. Let $\underline{x}$ be a sequences which at position n contains a block of $k+1$ zeros,

$$\underline{x} = \underbrace{\ldots\ldots 1}_{n}\underbrace{00\ldots 0}_{k+1}1\ldots$$

Then by shifting it n times we see that,

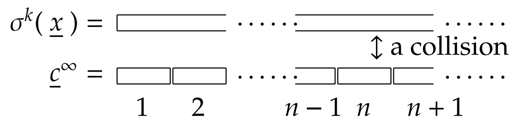
$$\sigma^n(\,\underline{x}\,) = \underbrace{00\ldots 0}_{k+1}1\ldots \;<\; \underbrace{00\ldots 0}_{k}1 = \underline{c_k}$$

and therefore it cannot be an element of $\mathcal{S}_{\underline{c_k}}$. Conversely, if $\underline{x}$ is a sequence that does not belong to $\mathcal{S}_{\underline{c_k}}$, then there is an integer n such that

$$\sigma^n(\,\underline{x}\,) < \underline{c_k}.$$

Hence the sequence $(\sigma^n(\underline{x}))(|\underline{c_k}|)$ must be smaller than $\underline{c_k}$, and therefore it must contain more that k consecutive zeros. $\square$

Let $\underline{c}$ be a sequences of finite length. It is clear that inclusion $\mathcal{S}_{\underline{c}^\infty} \subset \mathcal{S}_{\underline{c}}$ holds, as $\underline{c}$ is smaller than $\underline{c}^\infty$ and thereby allowing more sequences. If we let $\underline{x}$ be an element of $\mathcal{S}_{\underline{c}} \setminus \mathcal{S}_{\underline{c}^\infty}$ then there is a k such that $\sigma^k(\underline{x}) < \underline{c}^\infty$, otherwise it would be an element of $\mathcal{S}_{\underline{c}^\infty}$.



But then there is an integer n such that by shifting $n|\underline{c}| + k$ times we would have $\sigma^{n|\underline{c}|+k}(\underline{x}) < \underline{c}$, a contradiction. Hence we have proved the following lemma,

Lemma 3.2 *For a finite sequence $\underline{c}$ we have $\mathcal{S}_{\underline{c}^\infty} = \mathcal{S}_{\underline{c}}$.*

The above lemma says that $\mathcal{S}_{\underline{c}}$ does not change when $\underline{c}$ varies in the interval form $\underline{c}$ to $\underline{c}^\infty$. The question arising now is whether this interval can be extended, and if so, how much. The first step in answering this question is to introduce the central notion of minimal sequences.

Definition 3.3 *For a sequence $\underline{c}$ we define the integer $n_{\underline{c}}$, which may be not finite, by*

$$n_{\underline{c}} \;=\; \inf\{\, n \in \mathbb{N} : \underline{c}(n)^\infty \geq \underline{c}\,\}.$$

We say that $\underline{c}(n_{\underline{c}})$ is a minimal prefix of $\underline{c}$ and if $\underline{c}(n_{\underline{c}}) = \underline{c}$ we say that $\underline{c}$ is a minimal sequence or simply just minimal. We denote by $\mathcal{M}(n)$ the set of all minimal sequences of length precisely n.

Let us give some examples on this definition to illustrate some of the properties of minimal sequences.

Example 3.4 The minimal prefix of a sequence is the shortest prefix that when appended infinitely may times to itself majorices the origin sequences. To illustrate,

$$\begin{aligned} \underline{c} &= 00010010000111010\ldots \\ \underline{c}(n_{\underline{c}})^\infty &= \underbrace{0001001}_{n_{\underline{c}}}\underbrace{0001001}_{n_{\underline{c}}}\ldots \end{aligned}$$

Clearly, those sequences starting with 1 all have the minimal prefix 1, as the sequence with all ones dominates everything. □

Example 3.5 There are sequences such that their minimal prefix is not of finite length. To see this, consider the minimal sequences $\underline{c_1} = 01$, $\underline{c_2} = 01011$, $\underline{c_3} = 010110111$. With this idea in mind, we can construct a sequences with minimal prefix of any given length, i.e, we may take,

$$\underline{c} = 0101^2 01^3 01^4 \ldots, \tag{3.1}$$

to obtain a sequence without a minimal prefix of finite length, i.e. an infinite minimal sequence. □

Example 3.6 The set of all minimal sequences of length five is the set,

$$\mathcal{M}(5) = \{00001, 00011, 00101, 00111, 01011, 01111\}.$$

It is noticeable that all the elements of this set end with a 1, and that none of them contains an inner block of zeros exceeding the length of the first block of zeros. It is not hard to see that these two observations hold in general for finite minimal sequences. First, assuming that $\underline{c}(n_{\underline{c}})$ ends with a zero, and is not the trivial sequences 0, would lead to

$$\underline{c}(n_{\underline{c}})^{\infty} \leq \underline{c}(n_{\underline{c}} - 1)^{\infty},$$

as in the right-handside the ones are closer to the front of the sequence and therby making it larger. In the same way we see that if a minimal prefix contains a longer inner block of zeros, it will be bounded by the repetition of the new prefix created by breaking just before the block of zeros begins. That is, let $\underline{c}$ start with a block of k zeros and let it also later, at position n, have a block of $k+1$ zeros,

$$\begin{array}{rcl} \underline{c} & = & \overbrace{00\ldots0}^{k}1\ldots1\,\overbrace{00\ldots00}^{k+1}1\ldots \\ \underline{c}(n)^{\infty} & = & \underbrace{00\ldots01\ldots1}_{n}\,\underbrace{00\ldots01\ldots1}_{n}\ldots \end{array}$$

Clearly the repetition of the prefix $\underline{c}(n)$ dominates $\underline{c}$. This shows that having the strictly largest block of zeros at the front is a necessary condition for minimality. But as the sequence 01011 is minimal we see that it is not enough for obtaining a sufficient condition. □

Example 3.7 Let $\underline{c}$ be a finite minimal sequence. We can write $\underline{c}$ as a concatenation of sequences,

$$\underline{c} = \underline{c_1}\,\underline{c_2}\ldots\underline{c_k}\,\underline{d}\,,$$

with $|\underline{c_i}| = |\underline{c_1}|$ for $1 \le i \le k$, and $|\underline{d}| < |\underline{c_1}|$. As $\underline{c}$ is minimal it follows that

$$|\underline{c_i}| \;\ge\; |\underline{c_1}| \tag{3.2}$$

for $1 \le i \le k$. This as, if there is an m such that $|\underline{c_m}| < |\underline{c_1}|$ then we would have

$$\underline{c} = \underline{c_1}\,\underline{c_2}\ldots\underline{c_m}\ldots\underline{c_k}\,\underline{d} \;<\; \underline{c_1}\,\underline{c_2}\ldots\underline{c_{m-1}}\,\underline{c_1}.$$

Hence

$$(\,\underline{c}\,)^{\infty} \le (\underline{c_1}\,\underline{c_2}\ldots\underline{c_{m-1}})^{\infty},$$

which contradicts the minimality of $\underline{c}$. The inequality in (3.2) can not be improved to be sharp. This follows by considering the minimal sequence 01011 = (01)(01)1. □

In Example 3.6 we saw that the size of $\mathcal{M}(5)$ is 6. The size of $\mathcal{M}(n)$ grows fast when increasing n, ($\mathcal{M}(6) = 9$, $\mathcal{M}(7) = 18, \ldots,$ $\mathcal{M}(16) = 4080$). It is however not too difficult to find a lower estimate of the size of $\mathcal{M}(n)$ for large n, which indicates the speed of growth.

Proposition 3.8 *For n large we have*

$$|\mathcal{M}(n)| \geq 2^{n-2\sqrt{n}}.$$

Proof: Consider the sequence $\underline{x}$, of length n that starts with k consecutive zeros and at each $ki+1$ position it has a 1. That is,

$$\underline{x} = \underbrace{00\ldots0}_{k}1\underbrace{\ \ldots\ }_{k-1}1\underbrace{\ \ldots\ }_{k-1}1\ldots\ldots1\underbrace{\ \ldots\ }_{k-1}1$$

The gaps between the ones can now be filled with any sequences, and the sequence $\underline{x}$ will be minimal for them all, as having the strictly largest block of zeros at the front. The total number of such sequences $\underline{x}$ is at least

$$2^{n-k-n/k}.$$

This is clearly maximised when choosing $k = \sqrt{n}$, and the estimate follows. $\square$

In Example 3.5 we saw that there are infinite minimal sequences. This motivates the following notation.

Definition 3.9 *Denote by $\mathcal{M}$ the set of all infinite minimal sequences.*

Example 3.10 The sequence in Example 3.5 has a sub-sequence of strictly increasing blocks of ones. This is however not a sufficient condition of saying that the sequences is minimal. By taking the sequence in (3.1) and insert an extra zero after the first 1 we have

$$\underline{d} = 01001^201^301^4\ldots$$

The minimal prefix of this new sequence is of length 2, the sequences 01.

Even more, it is not true that an infinite minimal sequences has a sub-sequences of strictly increasing blocks of ones. Consider the infinite sequence

$$\underline{c} = 001(01)^\infty = 00101010101\ldots$$

By the argument in Example 3.6 this is an infinite minimal sequences, as its longest block of zeros is at the front. □

The next lemma gives us a way of characterising infinite minimal sequences via the left-shift.

Lemma 3.11 *An infinite sequence $\underline{c}$ is minimal if and only if we for all $n > 0$ have*

$$\sigma^n(\underline{c}) > \underline{c}\,. \tag{3.3}$$

Proof: Assume that $\underline{c}$ has a finite minimal prefix, $\underline{c}(n_{\underline{c}})$. Then by the definition of minimal prefix we have $\underline{c}(n_{\underline{c}})^\infty \geq \underline{c}$. There is a smallest N such that $\underline{c}(n_{\underline{c}})^N > \underline{c}$, since otherwise we would have $\underline{c}(n_{\underline{c}})^\infty = \underline{c}$, which would give that we can not have (3.3) for $n = n_{\underline{c}}$. By the existence of N we can for some sequence $\underline{d}$ write

$$\underline{c} = \underline{c}(n_{\underline{c}})^{N-1}\underline{d}\,,$$

with $\underline{d} < \underline{c}(n_{\underline{c}})$. By shifting $n_{\underline{c}}$ times we have

$$\sigma^{n_{\underline{c}}}(\underline{c}) = \underline{c}(n_{\underline{c}})^{N-2}\underline{d} \;<\; \underline{c}(n_{\underline{c}})^{N-1}\underline{d} = \underline{c}\,,$$

which contradicts (3.3). Conversely, if we for some n have equality or the reversed inequality in (3.3) then we have $\underline{c} \leq \underline{c}(n)^\infty$, and it follows that $\underline{c}$ has a finite minimal prefix. □

The above lemma implies that we may identify $\mathcal{M}$ by the set of sequences fulfilling the condition (3.3) for all $n > 0$, that is

$$\mathcal{M} = \{\, \underline{x} \in \Sigma_2 :\ \sigma^n(\underline{x}) > \underline{x}\,,\ \text{for all } n \geq 1 \,\}. \tag{3.4}$$

As will bee seen shortly this is a set of measure zero and has full dimension. For proving this, we shall make use of Birkhoff´s classical ergodic theorem, see [5] or [12] for a proof of Birkhoff's theorem. Recall that a transformation T is a measure invariant transformation if for a measure μ we have $\mu(T^{-1}(A)) = \mu(A)$, for any measurable set A with measurable pre-image. The Birkohoff ergodic theorem states:

Theorem 3.12 (Birkhoff) *Let $T : (X, \mu) \to (X, \mu)$ be a measure-preserving transformation of a probability space, $f \in L^1(X, \mu)$. Then for μ-almost every $x \in X$ the following time average exists:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m x)) = \int_X f(x)\, d\mu.$$

Theorem 3.13 *The set $\mathcal{M}$ has Lebesgue measure zero.*

Proof: On the unit interval the Lebesgue measure is clearly a probability measure. Recall the connection from (2.1) between the real unit interval and the set Σ_2. By that we can relate sequences to real numbers $\underline{c} \leftrightarrow c$.

Let T be the measure-preserving expansion map $2x \pmod 1$, which corresponds to the left-shift of sequences. Then the condition (3.3), that a sequences lacks a minimal prefix of finite length, can be translate into

$$\chi_{[0,c]}(T^m c) = 0, \qquad \text{for all } m \geq 1,$$

where χ is the characteristic function. By Birkhoff's Theorem 3.12 we have for Lebesgue almost every c

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \chi_{[0,c]}(2^m c) = \int_{[0,c]} dx = c.$$

Hence the set of c:s whose orbit under T never enters the interval $[0, c]$ must have measure 0. □

The proof of theorem actually gives a result stronger than the stated one. It says that even the set of sequences $\underline{c}$ such that

$$\sigma^n(\underline{c}) > \underline{c}$$

fails infinitely often, but sufficiently rarely, has measure zero.

By a slight reinterpretation of the set of infinite minimal sequences the next theorem is an almost direct consequence of Theorem 2.12.

Theorem 3.14 *The set $\mathcal{M}$ has full Hausdorff dimension,* $\dim_H \mathcal{M} = 1$.

Proof: Denote by $\mathcal{M}'$ the set of sequences form $\mathcal{M}$ where we have swapped the zeros to ones end ones to zeros, $0 \leftrightarrow 1$. We denote by $\underline{x}' \in \mathcal{M}'$ the element $\underline{x} \in \mathcal{M}$ where the elements are swapped. The swapping procedure will reverse the lexicographical order, that is if $\underline{x} < \underline{v}$ then $\underline{x}' > \underline{v}'$. Hence by (3.4) we may write

$$\mathcal{M}' = \{\, \underline{x} \in \Sigma_2 : \ \sigma^n(\,\underline{x}\,) < \underline{x}\,, \text{ for all } n \geq 1 \,\}.$$

Theorem 2.11 says that $\mathcal{M}'$ is the set of β-expansions of 1 for $1 < \beta < 2$. Let β_k be the real number related to the sequence 1^k. It is clear that

$$I(\beta_{k-1}, \beta_k) \subset \mathcal{M}'.$$

(Recall the definition of the set I form Section 2.2). By Theorem 2.12 and Proposition 2.2 we have

$$\begin{aligned}
\dim_H I(\beta_{k-1}, \beta_k) &= \dim_H \rho^{-1}([\beta_{k-1}, \beta_k]) \\
&\geq \frac{\ln \beta_{k-1}}{\ln \beta_k} \dim_H [\beta_{k-1}, \beta_k] \\
&\geq \frac{\ln \beta_{k-1}}{\ln \beta_k}.
\end{aligned}$$

By choosing k sufficiently large the right-handside gets arbitrarily close to 1, and the theorem follows. □

Theorem 3.13 indicates that from now-on we may assume that a sequence has a finite minimal prefix. We continue by studying and proving some additional properties of finite minimal prefixes.

Example 3.15 As already seen the first entry in a minimal sequence is always smaller than the last one. If we let

$$\underline{c} = 001001100101,$$

which is minimal then we see that it begins with a zero and ends with a 1. Even more, it starts with 00, which is smaller than the ending 01. If we look at all prefixes, of length $0 < k < 12$, and compare them to the suffixes of the same length we have

$$\underline{c}(k) \;=\; \left\{\begin{array}{lcr} 0 & & 1 \\ 00 & & 01 \\ 001 & & 101 \\ 0010 & & 0101 \\ 00100 & & 00101 \\ 001001 & < & 100101 \\ 0010011 & & 1100101 \\ 00100110 & & 01100101 \\ 001001100 & & 001100101 \\ 0010011001 & & 1001100101 \\ 00100110010 & & 01001100101 \end{array}\right\} \;=\; \underline{c}(n_{\underline{c}} - k + 1, n_{\underline{c}})$$

and we see that all the prefixes are smaller than their corresponding suffix. ◻

In the following lemma it is shown that for a finite minimal sequence $\underline{c}$ any prefix of length strictly less then $|c|$ is smaller than the suffix of the same length, as seen in the above example. The lemma is a bit technical, as it turns out to be several cases needed consider, but it is of great importance and will play a central role for deriving contradictions when proving forthcoming lemmas and theorems.

Lemma 3.16 *For any positive integer $0 < k < n_{\underline{c}}$ we have the strict inequality*

$$\underline{c}(k) \;<\; \underline{c}(n_{\underline{c}} - k + 1, n_{\underline{c}}).$$

Proof: For $k \leq n_{\underline{c}}/2$, let $\alpha = \underline{c}(k)$ and $\gamma = \underline{c}(n_{\underline{c}} - k + 1, n_{\underline{c}})$. Then we can write for some sequence β

$$\underline{c}(n_{\underline{c}}) = \alpha\beta\gamma.$$

Assume for contradiction that $\alpha \geq \gamma$. If the assumed inequality is strict, that is, $\alpha > \gamma$ then we would have,

$$\underline{c}(n_{\underline{c}})^{\infty} = (\alpha\beta\gamma)^{\infty} < \alpha\beta\alpha < (\alpha\beta)^{\infty},$$

which contradicts the definition of $n_{\underline{c}}$, as this gives a shorter prefix bounding $\underline{c}$. Therefore we shall from now-on assume that $\alpha = \gamma$.

First consider the case when α and β are of the same length, that is $|\alpha| = |\beta|$, (recall that we denote by $|\cdot|$ the length of a sequence). If $\alpha \geq \beta$ then

$$\underline{c}(n_{\underline{c}})^\infty = (\alpha\beta\alpha)^\infty \leq \alpha^\infty,$$

which again gives a contradiction to the minimality of $\underline{c}(n_{\underline{c}})$. Similarly we would have, if $\alpha < \beta$,

$$\underline{c}(n_{\underline{c}})^\infty = (\alpha\beta\alpha)^\infty = (\alpha\beta\alpha\alpha\beta\alpha)^\infty < (\alpha\beta)^\infty.$$

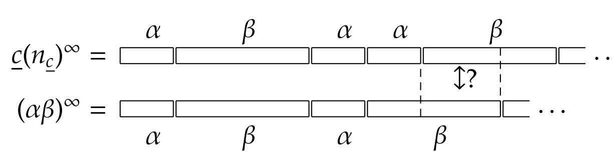
Up to this point we have only been comparing sequences of equal length to each-other, which turns out to give no complications. If we now start to consider the case when $|\alpha| < |\beta|$, we, besides the trivial cases, have to deal with the special case when α might be a prefix of β. Let's start with the trivial cases. If $\alpha \geq \beta$ it would follow that

$$\underline{c}(n_{\underline{c}})^\infty = (\alpha\beta\alpha)^\infty \leq \alpha^\infty.$$

And similarly if $\alpha < \beta$ and α is not a prefix of β,

$$\underline{c}(n_{\underline{c}})^\infty = (\alpha\beta\alpha)^\infty = (\alpha\beta\alpha\alpha\beta\alpha)^\infty < (\alpha\beta)^\infty. \tag{3.5}$$

The remaining case is now when α is a prefix of β. In (3.5) it is the third α-sequence that is compared to a β-sequence to achieve the inequality. It is here where the problem occurs. If α is a prefix of β this comparison fails, as what is needed is to compare a suffix of β to a prefix of β.



To get around the problem, make the implicit definition of the sequence β_1 by factoring out all the α-prefixes

$$\beta = \alpha^n \beta_1.$$

This only partly solves the problem as we soon might end up in the situation where β_1 is a prefix of α. Therefore we, in the same way as above implicitly define the sequence α_1 by

$$\alpha = \beta_1^m \alpha_1.$$

These prefix arguments can now be recursively repeated, that is, α_1 might be a prefix of β_1, and thereby we have to continue to define new sequences. By doing so we have defined a process by taking shorter and shorter prefixes. As we are dealing with sequences of finite length, and in each step of the process we factor out a sequence of positive length the process must end after a finite number of steps.

As the process alternates between factoring a β-sequence and an α-sequence we have to consider the two cases where the process ends after an even or odd number of steps.

First assume that the process ends after an even number of steps. We then have, by retracing backwards, for some sequences Δ_α and Δ_β

$$\begin{aligned} \alpha &= \beta_k^{m_k} \alpha_k \, \Delta_\alpha \, \alpha_k, \\ \beta &= \beta_k^{m_k} \alpha_k \, \Delta_\beta \, \beta_k. \end{aligned} \tag{3.6}$$

Using this factorisation we see that if $\alpha_k \leq \beta_k$ we would clearly have

$$\underline{c}(n_{\underline{c}})^\infty = ((\beta_k^{m_k} \alpha_k \, \Delta_\alpha \, \alpha_k)(\beta_k^{m_k} \alpha_k \, \Delta_\beta \, \beta_k)(\beta_k^{m_k} \alpha_k \, \Delta_\alpha \, \alpha_k))^\infty \leq (\beta_k)^\infty,$$

since α_k is not a prefix of β_k and then $\alpha_k \omega \leq \beta_k$ for any sequence ω. And if we would have $\alpha_k > \beta_k$ then notice that in the process the α_k's are separated by at least $\beta_k^{m_k}$. This gives

$$\underline{c}(n_{\underline{c}})^\infty = ((\beta_k^{m_k} \alpha_k \, \Delta_\alpha \, \alpha_k)(\beta_k^{m_k} \alpha_k \, \Delta_\beta \, \beta_k)(\beta_k^{m_k} \alpha_k \, \Delta_\alpha \, \alpha_k))^\infty \leq (\beta_k^{m_k} \alpha_k)^\infty,$$

which concludes the case of even steps.

If the process would end after an odd number of steps we have, similar to (3.6), fro some sequence Δ_α and Δ_β not necessarily that same as the ones above,

$$\alpha = \alpha_k^{n_k}\beta_{k+1}\,\Delta_\alpha\,\alpha_k,$$

$$\beta = \alpha_k^{n_k}\beta_{k+1}\,\Delta_\beta\,\beta_{k+1}.$$

If $\alpha_k \geq \beta_{k+1}$ then, again as above

$$\underline{c}(n_{\underline{c}})^\infty = ((\alpha_k^{n_k}\beta_{k+1}\,\Delta_\alpha\,\alpha_k)(\alpha_k^{n_k}\beta_{k+1}\,\Delta_\beta\,\beta_{k+1})(\alpha_k^{n_k}\beta_{k+1}\,\Delta_\alpha\,\alpha_k))^\infty \leq (\alpha_k)^\infty.$$

And for the case $\alpha_k < \beta_{k+1}$ then again notice that the process separates the β_{k+1}'s by at least $\alpha_k^{n_k}$, and therefore we have

$$\begin{aligned}\underline{c}(n_{\underline{c}})^\infty &= ((\alpha_k^{n_k}\beta_{k+1}\,\Delta_\alpha\,\alpha_k)(\alpha_k^{n_k}\beta_{k+1}\,\Delta_\beta\,\beta_{k+1})(\alpha_k^{n_k}\beta_{k+1}\,\Delta_\alpha\,\alpha_k))^\infty\\ &\leq (\alpha_k^{n_k}\beta_{k+1})^\infty,\end{aligned}$$

which concludes the case of odd numbers in the process and thereby also concludes the case of $|\alpha| < |\beta|$.

Next, consider the case of $|\alpha| > |\beta|$. Trivially if $\alpha \leq \beta$ it would follow that

$$\underline{c}(n_{\underline{c}})^\infty = (\alpha\beta\alpha)^\infty = (\alpha\beta\alpha\alpha\beta\alpha)^\infty \leq (\alpha\beta)^\infty.$$

Now if $\alpha > \beta$ we again run into the problem of having prefixes to deal with. If, first, β is not a prefix of α we would have

$$\underline{c}(n_{\underline{c}})^\infty = (\alpha\beta\alpha)^\infty < \alpha^\infty.$$

So, if β is a prefix of α it will lead to a comparison between a suffix of α to a prefix of α, all similar to the previous case. In the same way we construct the process of taking prefixes, starting by

$$\alpha = \beta^m\alpha_1,$$

and then by the same arguments as above we might have to continue the procedure and define β_1 by

$$\beta = \alpha_1^n \beta_1.$$

Continuing this process leads to, if ending after an even number of steps, again for some sequences Δ_α and Δ_β,

$$\alpha = \alpha_k^{n_k} \beta_k \, \Delta_\alpha \, \alpha_k,$$

$$\beta = \alpha_k^{n_k} \beta_k \, \Delta_\beta \, \beta_k.$$

If $\alpha_k \geq \beta_k$ then we clearly would have

$$\underline{c}(n_{\underline{c}})^\infty = ((\alpha_k^{n_k} \beta_k \, \Delta_\alpha \, \alpha_k)(\alpha_k^{n_k} \beta_k \, \Delta_\beta \, \beta_k)(\alpha_k^{n_k} \beta_k \, \Delta_\alpha \, \alpha_k))^\infty \leq (\alpha_k)^\infty.$$

For the case $\alpha_k < \beta_k$, we notice again that the process separates the β_k's by at least $\alpha_k^{n_k}$, and therefore

$$\underline{c}(n_{\underline{c}})^\infty = ((\alpha_k^{n_k} \beta_k \, \Delta_\alpha \, \alpha_k)(\alpha_k^{n_k} \beta_k \, \Delta_\beta \, \beta_k)(\alpha_k^{n_k} \beta_k \, \Delta_\alpha \, \alpha_k))^\infty \leq (\alpha_k^{n_k} \beta_k)^\infty.$$

If the process would have ended in an odd number of steps we get the factorisation, yet again for some sequences Δ_α and Δ_β,

$$\alpha = \beta_k^{m_k} \alpha_{k+1} \, \Delta_\alpha \, \alpha_{k+1},$$

$$\beta = \beta_k^{m_k} \alpha_{k+1} \, \Delta_\beta \, \beta_k.$$

And as in the previous cases, if $\alpha_{k+1} \leq \beta_k$ we would have

$$\begin{aligned} \underline{c}(n_{\underline{c}})^\infty &= ((\beta_k^{m_k} \alpha_{k+1} \, \Delta_\alpha \, \alpha_{k+1})(\beta_k^{m_k} \alpha_{k+1} \, \Delta_\beta \, \beta_k)(\beta_k^{m_k} \alpha_{k+1} \, \Delta_\alpha \, \alpha_{k+1}))^\infty \\ &\leq (\beta_k)^\infty, \end{aligned}$$

and finally we would by same arguments as before about separation have

$$\begin{aligned} \underline{c}(n_{\underline{c}})^\infty &= ((\beta_k^{m_k} \alpha_{k+1} \, \Delta_\alpha \, \alpha_{k+1})(\beta_k^{m_k} \alpha_{k+1} \, \Delta_\beta \, \beta_k)(\beta_k^{m_k} \alpha_{k+1} \, \Delta_\alpha \, \alpha_{k+1}))^\infty \\ &\leq (\beta_k^{m_k} \alpha_{k+1})^\infty \end{aligned}$$

for the case $\alpha_{k+1} > \beta_k$, and completes the case with $k \leq n_{\underline{c}}/2$. We postpone the proof of the case $n_{\underline{c}}/2 < k < n_{\underline{c}}$ until page 36. □

We may postpone the rest of the proof without falling into any trouble, as what we are going to use in the proofs of the next lemmas will fall into the case of what we have proved.

Example 3.17 The minimal prefix of the sequence

$$\underline{c} = 00010010000111010$$

is the sequence $\underline{c}(n_{\underline{c}}) = 0001001$. By taking the minimal prefix of $\underline{c}(n_{\underline{c}})$ we see that the minimal prefix is the prefix itself. □

The above example illustrates a result that holds in general, and is given in the following lemma.

Lemma 3.18 *For any positive integer $m < n_{\underline{c}}$ we have $\underline{c}(m)^\infty < \underline{c}(n_{\underline{c}})$.*

Proof: Define by $k_{\underline{c}} = n_{\underline{c}(n_{\underline{c}})}$, that is, $k_{\underline{c}}$ is the smallest integer such that $\underline{c}(k_{\underline{c}})^\infty \geq \underline{c}(n_{\underline{c}})$. Assume for contradiction that $k_{\underline{c}} < n_{\underline{c}}$. As $\underline{c}(n_{\underline{c}})$ is of finite length there is a smallest positive integer N such that $\underline{c}(k_{\underline{c}})^N \geq \underline{c}(n_{\underline{c}})$. The case of equality can be out-ruled, since otherwise $\underline{c}(n_{\underline{c}})$ would be a repetition of $\underline{c}(k_{\underline{c}})$, which contradicts the definition of $n_{\underline{c}}$. Hence we have

$$\underline{c}(k_{\underline{c}})^N \;>\; \underline{c}(n_{\underline{c}}). \tag{3.7}$$

Again by the definition of $n_{\underline{c}}$ it follows that N must be chosen so that $Nk_{\underline{c}}$ is strictly larger than $n_{\underline{c}}$, since otherwise we would have found a shorter bounding prefix of $\underline{c}$. On the other hand, as any minimal prefix ends with a 1, it follows that this is also a sufficient choice of N. Hence we have

$$k_{\underline{c}}(N-1) \;<\; n_{\underline{c}} \;<\; k_{\underline{c}}N.$$

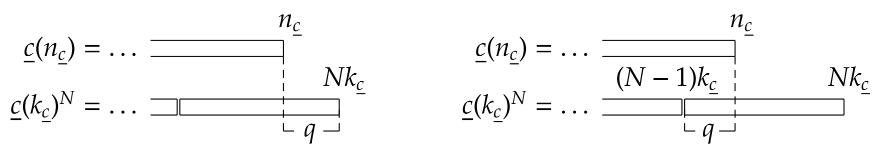
As $\underline{c}(n_{\underline{c}})$ is the minimal prefix of $\underline{c}$ we have $\underline{c}(k_{\underline{c}})^\infty < \underline{c}(n_{\underline{c}})^\infty$. This implies the inequality, by considering prefixes,

$$\left(\underline{c}(k_{\underline{c}})^N\right)(n_{\underline{c}}) \;\leq\; \underline{c}(n_{\underline{c}}). \tag{3.8}$$

Combining (3.7) and (3.8) shows that $\left(\underline{c}(k_{\underline{c}})^N\right)(n_{\underline{c}})$ must be equal to $\underline{c}(n_{\underline{c}})$. Now define the number q by

$$q \;=\; \min\left\{\, Nk_{\underline{c}} - n_{\underline{c}},\; n_{\underline{c}} - (N-1)k_{\underline{c}} \,\right\}.$$

This number is the length of the smallest of the parts, of the last repetition of $\underline{c}(k_{\underline{c}})$, that either extends beyond or overlaps $\underline{c}(n_{\underline{c}})$



Denote by $\alpha = \underline{c}(q)$ and $\gamma = \underline{c}(k_{\underline{c}} - q + 1, k_{\underline{c}})$. We can then for some sequence β write $\underline{c}(k_{\underline{c}}) = \alpha\beta\gamma$. By the equality in the first positions it follows that then $\underline{c}(n_{\underline{c}})$ must be of the form, depending on the choice in the definition of q,

$$\underline{c}(n_{\underline{c}}) = \begin{cases} (\alpha\beta\gamma)^{N-1}\alpha & \text{if } n_{\underline{c}} - k_{\underline{c}}(N-1) < Nk_{\underline{c}} - n_{\underline{c}}, \\ (\alpha\beta\gamma)^{N-1}\alpha\beta & \text{otherwise.} \end{cases}$$

But this is impossible as Lemma 3.16 would imply $\alpha < \alpha$, or $\alpha\beta < \alpha\beta$, a contradiction. □

If reformulating, the above lemma says that a minimal prefix is a minimal sequences. In our notation that is

$$n_{\underline{c}} = n_{\underline{c}(n_{\underline{c}})} = n_{\underline{c}(n_{\underline{c}(n_{\underline{c}})})} = \ldots .$$

Next we turn to the question of characterising the minimal sequences via the left-shift. In Lemma 3.11 we saw how we can characterise the infinite minimal sequences via the left-shift. We shall shortly see that we can do a similar characterisation of the finite minimal sequences. We start with a lemma before giving the main theorem settling this characterisation question.

Lemma 3.19 *A finite sequence $\underline{c}$ is periodic with a minimal sequence as period if and only if*

$$\underline{c}^{\infty} \in \mathcal{S}_{\underline{c}}. \tag{3.9}$$

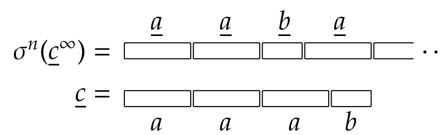
Proof: Let $\underline{c}$ be a minimal sequence. Assume for contradiction that there is an $n \leq |\underline{c}|$ such that

$$\sigma^{n}(\,\underline{c}^{\infty}\,) < \underline{c}. \tag{3.10}$$

If $n \leq |\underline{c}|/2$, let $\underline{a} = \underline{c}(n)$. Then we can for some maximal integer N and a sequence $\underline{b}$ write

$$\underline{c} = \underline{a}^{N}\underline{b}.$$

We must have $\underline{b} \leq \underline{a}$, as $\underline{b} > \underline{a}$ would contradict (3.10).



This leads to two cases needing consideration, where the first one is; $|\underline{b}| \leq |\underline{a}|$. The sequence $\underline{b}$ can not be void as then $\underline{a}$ would be periodic, contradicting the minimality of $\underline{c}$. Therefore let $\underline{d}$ be the prefix of $\underline{a}$ with the same length as $\underline{b}$. Then Lemma 3.16 gives that $\underline{d} < \underline{b}$, contradicting $\underline{a} > \underline{b}$. Secondly, if $|\underline{b}| > |\underline{a}|$ then we would have

$$\underline{c}^{\infty} = (\underline{a}^{N}\underline{b})^{\infty} \leq \underline{a}^{\infty}$$

contradicting the minimality of $\underline{c}$.

If $n > |\underline{c}|/2$, let $\underline{a} = \underline{c}(|c| - n)$ and $\underline{b} = \underline{c}(n+1, |c|)$. Then by our assumption we must have $\underline{a} \geq \underline{b}$, which contradicts the reverse inequality $\underline{a} < \underline{b}$ implicated by Lemma 3.16. By Lemma 3.2 it follows that for each $k \in \mathbb{N}$ we have

$$(\underline{c}^{\,k})^{\infty} \in \mathcal{S}_{\underline{c}^{k}}.$$

Conversely, assume $\underline{c}$ is not periodic with a minimal sequence as period. Then $\underline{c}$ is of the form $\underline{c} = \underline{a}^{N}\underline{b}$, where $\underline{a}$ is the minimal prefix and N is the largest possible integer. We have $\underline{b} < \underline{a}$, otherwise $\underline{c}$ would be periodic or $\underline{a}$ wouldn't be the minimal prefix. Let $n = |\underline{a}|^N$. If $|\underline{b}| \geq |\underline{a}|$ then clearly

$$\sigma^n(\underline{c}^\infty) = \sigma^n((\underline{a}^N\underline{b})^\infty) = (\underline{b}\underline{a}^N)^\infty < \underline{a}^N\underline{b} = \underline{c}, \tag{3.11}$$

and therefore we can not have (3.9). If $|\underline{b}| < |\underline{a}|$ and $\underline{b}$ is not a prefix of $\underline{a}$ then we again obtain the above inequality (3.11). If, finally, $|\underline{b}| < |\underline{a}|$ and $\underline{b}$ is a prefix of $\underline{a}$, write $\underline{a}$ as the concatenation

$$\underline{a} = \underline{a_1}\,\underline{a_2}\,\underline{a_3}\dots\underline{a_k}\,\underline{d}$$

where $|\underline{a_i}| = |\underline{b}|$ for all $1 \leq i \leq k$ and $0 \leq |\underline{d}| < |\underline{b}|$. As $\underline{b}$ is a prefix of $\underline{a}$ we have $\underline{a_1} = \underline{b}$. By the minimality of $\underline{a}$ we have $\underline{a_1} \leq \underline{a_2}$, that is $\underline{b} \leq \underline{a_2}$. If we are in the case $\underline{b} < \underline{a_2}$ it follows that

$$\begin{aligned}
\sigma^n((\underline{a}^N\underline{b})^\infty) &= (\underline{b}\underline{a}^N)^\infty \\
&= \underline{b}\,\underline{b}\,\underline{a_2}\,\underline{a_3}\dots\underline{a_k}\,\underline{d}\,\underline{a}^{N-1}(\underline{b}\underline{a}^N)^\infty \\
&< \underline{b}\,\underline{a_2}\,\underline{a_3}\dots\underline{a_k}\,\underline{d}\,\underline{a}^{N-1} \\
&< \underline{a}^N\underline{b},
\end{aligned}$$

and again we see that (3.9) does not hold. If $\underline{b} = \underline{a_2}$ then use the same arguments on how $\underline{a_3}$ is related to $\underline{b}$. Continuing this way as long as necessary we will find $\underline{b} < \underline{a_i}$ for some i or that all the $\underline{a_i}$'s are the same. But if we are in the latter case and $|\underline{d}| = 0$, then Lemma 3.16 says that we must have $\underline{a_1} < \underline{a_k}$ and we can do the same estimate as before. If, on the other-hand $\underline{d}$ is non void, let $\underline{e}$ be the prefix of $\underline{a_1}$ with the same length as $\underline{d}$. Again as $\underline{a}$ is a minimal sequence Lemma 3.16 gives $\underline{e} < \underline{d}$ and therefore $\underline{b} < \underline{d}$. This implies

$$\sigma^n((\underline{a}^N\underline{b})^\infty) = (\underline{b}\underline{a}^N)^\infty = \underline{b}\,\underline{b}^k\underline{d}\,\underline{a}^{N-1}(\underline{b}\underline{a}^N)^\infty < \underline{b}^k\underline{d}\,\underline{a}^{N-1} < \underline{a}^N\underline{b},$$

which concludes the proof. □

Let us give a very short example illustrating the statement in the above lemma.

Example 3.20 It is clear that we have $(01)^\infty \in \mathcal{S}_{01}$, but also $(0101)^\infty \in \mathcal{S}_{0101}$. □

We now return to the completion of the proof of Lemma 3.16, which is an almost direct consequence of the above result. Again, we wish to point out that the proof of Lemma 3.19 does not use Lemma 3.16 to compare prefix to suffixes in the case where they overlap.

Completion of the proof of Lemma 3.16: For the case $n_{\underline{c}}/2 < k < n_{\underline{c}}$, we put

$$\underline{c}(n_{\underline{c}}) = \alpha\beta\gamma,$$

where $|\alpha| = |\gamma| = n_{\underline{c}} - k$ and thereby $|\beta| = 2k - n_{\underline{c}}$. What we need to show is that the inequality $\alpha\beta < \beta\gamma$ holds. By Lemma 3.19 we have

$$\alpha\beta\gamma \leq \beta\gamma\alpha.$$

If we have the equality $\alpha\beta = \beta\gamma$ then the first part of the proof of this lemma would give us that $\beta\gamma\alpha < \alpha\beta\gamma$, as $\alpha < \gamma$. Hence we must have $\alpha\beta < \beta\gamma$. □

We have now reached the point where we can state and prove the reaming parts of the main theorem of characterising the minimal sequences via the left-shift.

Theorem 3.21 *An infinite sequence $\underline{c}$ is minimal if and only if*

$$\sigma^n(\underline{c}) > \underline{c} \quad \text{for all } n > 0.$$

A finite sequence $\underline{c}$ is minimal if and only if

$$\sigma^n(\underline{c}) > \underline{c} \quad \text{for all } |c| > n > 0. \tag{3.12}$$

Proof: The first part of the theorem is Lemma 3.11. In the second part, the necessity is given by Lemma 3.16. For the sufficiency, notice that if (3.12) holds then $\underline{c}^\infty$ is an element in $\mathcal{S}_{\underline{c}}$ and hence by Lemma 3.19 $\underline{c}$ is periodic with a minimal sequence as period. That is, $\underline{c} = \underline{a}^k$, for a minimal sequences $\underline{a}$. But then if $k > 1$ we would have

$$\sigma^{|\underline{a}|}(\,\underline{c}\,) = \underline{a}^{k-1} < \underline{a}^k = \underline{c}\,,$$

contradicting (3.12). Hence $k = 1$, and therefore $\underline{c}$ is minimal. □

We end this section by introducing and stating some properties of the sequences $\underline{a_n}$ and $\underline{b_n}$. We define for a given finite sequence $\underline{c}$ and for the positive integer n the sequences $\underline{a_n}$ and $\underline{b_n}$ by

$$\begin{aligned} \underline{a_n} &= \underline{a_{n,\underline{c}}} = \underline{c}(n_{\underline{c}} - 1)\, 01^n, \\ \underline{b_n} &= \underline{b_{n,\underline{c}}} = \underline{c}(n_{\underline{c}})^n 1. \end{aligned} \tag{3.13}$$

It will be clear from the context on which sequence $\underline{c}$ the sequences $\underline{a_n}$ and $\underline{b_n}$ are based on. Notice that by this definition, $\underline{a_n}$ tends monotonously to $\underline{c}(n_{\underline{c}})$ from below and $\underline{b_n}$ tends monotonously to $\underline{c}(n_{\underline{c}})^\infty$ from above when letting n tend to infinity. Their property of approximating a minimal prefix will be of great importance, and they will play a crucial role in the proofs of the following sections.

Example 3.22 If we let $\underline{c} = 0010010001$, then the minimal prefix of $\underline{c}$ will be $\underline{c}(n_{\underline{c}}) = 001$. Hence we have

$$\underline{a_1} = 0001, \quad \underline{a_2} = 00011, \quad \underline{a_3} = 000111,$$

and more generally, $\underline{a_n} = 0001^n$. The $\underline{b_n}$ sequences will be

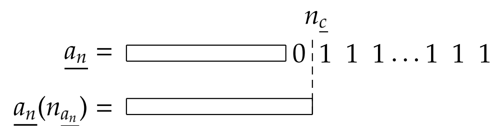
$$\underline{b_1} = 0011, \quad \underline{b_2} = 0010011 = (001)^2 1$$

$$\underline{b_3} = 0010010011 = (001)^3 1,$$

which clearly generalises to $\underline{b_n} = (001)^n 1$. □

Lemma 3.23 *There is an N such that the sequences $\underline{a_n}$ are minimal for $n > N$.*

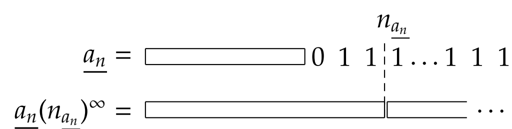
Proof: Assume for contradiction that $n_{\underline{a_n}}$ is less than $n_{\underline{c}}-1+n = |\underline{a_n}|$, that is, $\underline{a_n}$ is not minimal. If $n_{\underline{a_n}}$ would be equal to $n_{\underline{c}}$ it would contradict that any finite minimal prefix ends with a one.



If we assume that $n_{\underline{a_n}} > n_{\underline{c}}$ then we would have that

$$\underline{a_n} = \underline{c}(n_{\underline{c}} - 1)01^n > \underline{a_n}(n_{\underline{a_n}})^\infty$$

as the sequences are equal in the first $n_{\underline{a_n}}$ positions and then $\underline{a_n}$ has a one while $\underline{a_n}(n_{\underline{a_n}})^\infty$ a zero, a contradiction to the assumed minimality.



Finally, if $n_{\underline{a_n}} < n_{\underline{c}}$ we then have

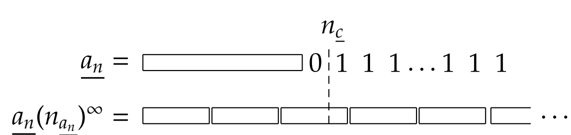
$$\underline{a_n} = \underline{c}(n_{\underline{c}} - 1)01^n \;\leq\; \underline{c}(n_{\underline{a_n}})^\infty = \underline{a_n}(n_{\underline{a_n}})^\infty. \tag{3.14}$$

But by Lemma 3.18 we have the inequality

$$\underline{c}(n_{\underline{c}}) = \underline{c}(n_{\underline{c}} - 1)1 > \underline{c}(n_{\underline{a_n}})^\infty \tag{3.15}$$

Combining (3.14) and (3.15) we see that when replacing the 0 at position $n_{\underline{c}}$ the inequality is reversed. This implies that $\underline{a_n}$ is equal to $\underline{a_n}(n_{\underline{a_n}})^\infty$ in the first $n_{\underline{c}}$ positions, that is

$$\underline{a_n}(n_{\underline{c}}) = \left(\underline{a_n}(n_{\underline{a_n}})^\infty\right)(n_{\underline{c}}).$$



If now N is chosen large enough then this implies that $\underline{a_n}(n_{\underline{a_n}})$ must be larger than a block of $n_{\underline{a_n}}$ ones, which is not possible. □

By looking at the procedure in the proof of the above lemma, we see that we can say, if $\underline{a_n}$ is minimal for some n then $\underline{a_{n+1}}$ is also minimal. That is we just have to find an n such that $\underline{a_n}$ and then we may conclude that all the $\underline{a_n}$'s are minimal for larger n.

Example 3.24 The statement in Lemma 3.23 that for some N the sequences $\underline{a_n}$ are minimal for $n > N$ can not be improved to be valid for all n. To see this, let $\underline{c} = 00101$, which is minimal and therefore $\underline{c}(n_{\underline{c}}) = 00101$. Then $\underline{a_1} = 001001$, which is not minimal, as it is a repetition of 001. But we have $\underline{a_2} = 0010011$, which is minimal.

Even more, for any k there is a sequences $\underline{c}$ such that $\underline{a_n}$ is not minimal for $n < k$. This since, letting $\underline{c} = 01^k$, which is minimal, we have

$$\begin{aligned}
\underline{a_n} &= 0\overbrace{1\ldots\ldots1}^{k-1}0\overbrace{1\ldots1}^{n} \\
\underline{a_n}(n_{\underline{a_n}})^\infty &= \underbrace{01\ldots\ldots1}_{n_{\underline{a_n}}}\underbrace{01\ldots\ldots1}_{n_{\underline{a_n}}}\ldots
\end{aligned}$$

This implies that $\underline{a_n}$ cannot be minimal until n surpasses k. □

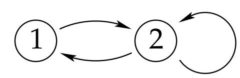
Notice that from the definition of the sequences $\underline{a_n}$ and by Lemma 3.23 it follows that each minimal sequences is an accumulation point of minimal sequences.

4. Symbolic Dynamics

A square matrix, with the entries 0 and 1, a zero-one matrix, corresponds to a directed graph. The entries in the matrix corresponds to whether there is an edge or not between two nodes, where the nodes in the graph gives the indices for rows and columns.

Example 4.1 A simple graph with a corresponding matrix,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

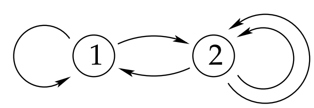


Note that the matrix representation is not unique. Changing the labels of the nodes in the graph will permute the rows and columns in the matrix. □

By extending the matrices to allow the entries to be natural numbers we have the interpretation that there is more than one edge between two nodes in the graph. That is, the entries tell how many ways there are to reach a node in. The entries in the square of a matrix say how many ways there are to reach a node in precisely two steps. This follows by considering the definition of matrix multiplication. This clearly generalises to higher powers of the matrix.

Example 4.2 The square of the matrix A in Example 4.1 will be

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$



The double loop at node 2 corresponds to the two paths $2 \to 1 \to 2$ and $2 \to 2 \to 2$ in A. □

For two matrices A and B, of the same size, we say that $A \geq B$ if and only if $(A)_{ij} \geq (B)_{ij}$ for all i, j. Similarly we say that $A > 0$ if and only if all the entries are positive.

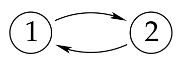
The matrices we are going to consider are matrices corresponding to graphs where all nodes are reachable in n steps for any n larger than some N. This is an important property which motivates the following definition.

Definition 4.3 *A square matrix A is said to be irreducible if there exists an integer n such that $A^n > 0$.*

By considering the standard matrix multiplication it follows quite immediately that if $A^n > 0$ for some n then also $A^{n+1} > 0$.

Example 4.4 There are graphs where all nodes can be reached, but never in the same number of steps. A simple example is the following graph with two nodes,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



It follows that for any odd power $A^{2k+1} = A$ and for all even powers the matrix collapses into the identity matrix. □

Let $\Sigma_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$ be the set of all one-sided infinite sequences of N symbols. Let A be an $N \times N$ be a zero-one matrix and let

$$\Sigma_A = \left\{ \underline{x} \in \Sigma_N : A_{x_n x_{n+1}} = 1 \text{ for } n \geq 1 \right\}.$$

It is clear that the matrix A determines all the admissible transitions between the symbols $\{0, 1, \ldots, N-1\}$. The set Σ_A is shift invariant. The matrix A is often called a *transition matrix*.

Definition 4.5 *The restriction*

$$\sigma|_{\Sigma_A} := \sigma_A$$

is called the topological Markov chain determined by the transition matrix A. Somtimes σ_A is also called a subshift of finite type.

By previous examples we see that the elements of Σ_A can bee seen as infinite paths in a directed graph.

Let T be the set of all binary sequences of length precisely k, and denote by f a bijection, $f : T \to \{0, 1, \ldots, 2^k - 1\}$. For an element $\underline{x}$ of Σ_2 we introduce here the notation

$$m_{\underline{x}}(n) = f\left(\underline{x}(n, n+k-1)\right).$$

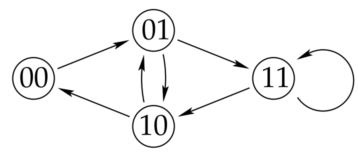
Then, similar to above, for A a $2^k \times 2^k$ zero-one matrix, we can define

$$\Sigma_{2,A} = \left\{ \underline{x} \in \Sigma_2 : A_{m_{\underline{x}}(n)\, m_{\underline{x}}(n+1)} = 1 \text{ for } n \geq 1 \right\}. \tag{4.1}$$

This set can also be represented by a graph, where the nodes are labelled with the sequences from T.

Example 4.6 A transition matrix A and a corresponding graph for node-sequences of length 2 is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$



The path

$$01 \to 11 \to 11 \to 10 \to 01 \to 10 \to 00 \to \ldots$$

corresponds to the element $\underline{x}$ of $\Sigma_{2,A}$

$$\underline{x} = 0\,1\,1\,1\,0\,1\,0\,0\,\ldots$$

It follows that for this transition matrix the set $\Sigma_{2,A}$ will be the subset of Σ_2 where the elements do not contain 3 consecutive zeros. In Example 3.1 we saw that this is precisely the set $\mathcal{S}_{\underline{c}}$ for $\underline{c} = 001$. □

As seen, a matrix give rise to a set of sequences. We would like to do the reversed, from a set of sequences find a matrix describing it. To do so, notice that for a fixed $\underline{c}$ the set $\mathcal{S}_{\underline{c}}(n_{\underline{c}} - 1)$ contains only finitely many elements, say N, as we are looking at sequences with finite minimal prefix. Let f denote a bijection $f : \mathcal{S}_{\underline{c}}(n) \to \{0, 1, \ldots, N-1\}$. Then we define the $N \times N$ zero-one matrix $A_{\underline{c}}$ by

$$(A_{\underline{c}})_{ij} = 1 \quad \text{if and only if} \quad \begin{cases} \text{There are } \underline{u} \text{ and } \underline{v} \text{ in } \mathcal{S}_{\underline{c}}(n_{\underline{c}} - 1) \\ \text{such that } \underline{u}(2, n_{\underline{c}} - 1) = \underline{v}(1, n_{\underline{c}} - 2), \\ \text{and } \underline{u}(1)\underline{v}(1, n_{\underline{c}} - 1) \geq \underline{c}(n_{\underline{c}}) \text{, where} \\ i = f(\underline{u}) \text{ and } j = f(\underline{v}). \end{cases} \tag{4.2}$$

Then by recalling (4.1), the so defined matrix give rise to the set $\Sigma_{2,A_{\underline{c}}}$. This set is actually the set $\mathcal{S}_{\underline{c}}$. To see this, notice first that the condition for having $\underline{x}$ as an element of the set $\mathcal{S}_{\underline{c}}$ for some $\underline{c}$ is equivalent to say that

$$\underline{x}(k, k + n_{\underline{c}} - 1) \; \geq \; \underline{c}(n_{\underline{c}}) \qquad \text{for all } k \geq 1, \tag{4.3}$$

that is, each sub-sequence of $\underline{x}$ of length $n_{\underline{c}}$ must be equal or larger than $\underline{c}(n_{\underline{c}})$.

It is clear that for any element $\underline{x}$ in $\Sigma_{2,A_{\underline{c}}}$ we have (4.3), which implies that $\underline{x}$ is also an element of $\mathcal{S}_{\underline{c}}$. Conversely, for any $\underline{x}$ in $\mathcal{S}_{\underline{c}}$ we have that any sub-sequence of $\underline{x}$ of length $n_{\underline{c}}$ is an element of the set of prefixes. This implies

$$\underline{x}(k, k + n_{\underline{c}} - 2) \; \in \mathcal{S}_{\underline{c}}(n_{\underline{c}} - 1) \qquad \text{for all } k \geq 1.$$

Hence we may find sequences $\underline{u}$ and $\underline{v}$ in $\mathcal{S}_{\underline{c}}(n_{\underline{c}} - 1)$ fulfilling the conditions in the definition (4.2) of the transition matrix $A_{\underline{c}}$. This gives the reversed inclusion, and therefore we may conclude

$$\Sigma_{2,A_{\underline{c}}} \; = \; \mathcal{S}_{\underline{c}}.$$

This motivates the following definition.

Definition 4.7 *The matrix associated to the graph describing the set of sequences $\mathcal{S}_{\underline{c}}$ via the $\mathcal{S}_{\underline{c}}(n_{\underline{c}} - 1)$ we call the transition matrix $A_{\underline{c}}$.*

Notice that in the definition of the transition matrix $A_{\underline{c}}$ it is not required that the sequence $\underline{c}$ is minimal. The transition matrix is defined via the minimal prefix of $\underline{c}$. That is, if two different sequences $\underline{c}$ and $\underline{d}$ have the same minimal prefix then they have the same transition matrix, $A_{\underline{c}} = A_{\underline{d}}$.

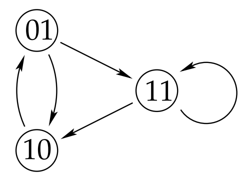
Lemma 4.8 *The transition matrix $A_{\underline{c}}$ is irreducible.*

We postpone the proof of the lemma until page 48, where some additional properties of the structure of $\mathcal{S}_{\underline{c}}$ have been introduced and derived.

In the definition of the transition matrices $A_{\underline{c}}$ we used the set $\mathcal{S}_{\underline{c}}(n_{\underline{c}}-1)$. The set $\mathcal{S}_{\underline{c}}$ can be coded via matrices of larger size if just considering the $\mathcal{S}_{\underline{c}}(k)$ for $k > n_{\underline{c}} - 1$. It will still give an irreducible matrix as no unreachable nodes in the graph has been added.

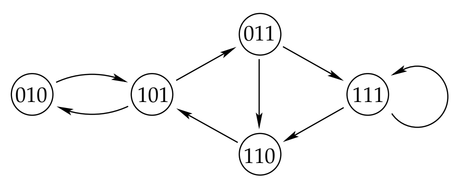
Example 4.9 The 2×2 transition matrix $A_{\underline{c}}$ for $\underline{c} = 01$ is, if relabelling the nodes, the matrix given in Example 4.1. By relabelling the nodes with longer sequences, we can code the same set with a larger matrix,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$



By relabelling a second time we get an even larger matrix,

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



□

The transition matrices can be recoded in such a way that they take any size larger than some N, but then the irreducibility may be lost. This occurs by introducing new nodes, which never will be connected in the graph.

A classical and often used theorem in symbolic dynamics is the Perron-Frobenius theorem. It will later on play a key-role to us. What we state here is a result due to Perron [18] and later generalised by Frobenius. We use the formulation given in Kitchens book on symbolic dynamics [14]. The theorem states:

Theorem 4.10 (Perron-Frobenius) *Suppose A is a nonnegative, square matrix. If A is irreducible there exists a real eigenvalue $\lambda > 0$ such that*

(i). λ is a simple root of the characteristic polynomial;

(ii). λ has strictly positive left and right eigenvectors;

(iii). the eigenvectors for λ are unique up to constant multiple;

(iv). $\lambda > |\mu|$, where μ is any other eigenvalue;

(v). if $0 \leq B \leq A$ and β is an eigenvalue for B then $|\beta| \leq \lambda$ and equality occurs if and only if $B = A$.

The special eigenvalue λ, is the Perron value of the matrix A. A positive eigenvector corresponding to λ is called a Perron eigenvector.

Notice here that the Perron value of an irreducible matrix A coincides with the spectral radius $\rho(A)$, which is

$$\rho(A) = \max_i |\lambda_i|.$$

where the λ_i are the eigenvalues to the matrix A.

The transition matrix $A_{\underline{c}}$ was defined to have a one if and only if there where a transition between two sequences of length $n_{\underline{c}} - 1$. This transition then describes a sequence of length $n_{\underline{c}}$. That is, the number of ones in the transition matrix is the number of sequences of length $n_{\underline{c}}$, written out, that is

$$\left|\mathcal{S}_{\underline{c}}(n_{\underline{c}})\right| = \sum_{i,j} (A_{\underline{c}})_{ij}$$

In Example 4.2, and in the discussion just before it, we saw that taking k-th power of the transition matrix, is the same as counting the number of nodes that can be reached in precisely k steps, which implies,

$$\left|\mathcal{S}_{\underline{c}}(n_{\underline{c}} + k)\right| = \sum_{i,j} (A_{\underline{c}}^k)_{ij},$$

Each power of the transition matrix can then be seen as appending to our starting sequence one symbol.

For an $m \times m$ matrix A we introduce the two matrix-norms $||\cdot||_1$ and $||\cdot||_2$ by

$$||A||_1 = \sum_{i,j} \left|A_{ij}\right|, \qquad ||A||_2 = \sqrt{\sum_{i,j} \left|A_{ij}\right|^2}$$

The second one is the standard Euclidian norm, which has the property

$$||A||_2^2 = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_m^2,$$

where the λ_i's are the eigenvalues of A. It follows by a straight forward calculation that, for some constant C, we have

$$\rho(A) \ \leq \ ||A||_1 \ \leq \ C\,||A||_2 \ \leq \ mC\,\rho(A).$$

Rewriting this, we may say that there exists constants k_1 and k_2 with, for n sufficiently large,

$$k_1\lambda^n \ \leq \ \left|\mathcal{S}_{\underline{c}}(n)\right| \ \leq \ k_2\lambda^n \tag{4.4}$$

where λ is the spectral radius of the transition matrix $A_{\underline{c}}$ The identity (4.4) also implies the useful result

$$h_{\text{top}}(\mathcal{S}_{\underline{c}}) \ = \ \log \rho(A_{\underline{c}}) \tag{4.5}$$

for calculating the topological entropy.

Example 4.11 Consider the two transition matrices

$$A_{001} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_{011} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

By introducing an extra node, (an unreachable one), in the graph corresponding to the matrix A_{011} we obtain the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

It is clear that $A_{001} \geq B$ with $(A)_{12} > (B)_{12}$ and as $A_{001}^3 > 0$, it is irreducible. The Perron-Frobenius Theorem 4.10 says that A_{001} has a strictly larger largest eigenvalue than the matrix B, that is

$$\rho(A_{001}) = 1.839 \; > \; 1.466 = \rho(B).$$

and from (4.5) we have the topological entropy

$$h_{\text{top}}(\mathcal{S}_{001}) = \log \rho(A_{001}) = \log 1.839.$$

The topological entropy of $\mathcal{S}_{011}$ is can be found via the matrix A_{011} as before, or via the matrix B. This is because the inserted extra node when obtaining B was a non-sequences generating node, that is, it does not allow more words. Hence

$$h_{\text{top}}(\mathcal{S}_{011}) = \log \rho(A_{011}) = \log \rho(B) = \log 1.466.$$

□

5. Proof of the theorems

We have now achieved a sufficient basis for proving our remaining main results for characterising the map ϕ. By Lemma 3.2 we have that ϕ is constant on intervals $[\underline{c}(n_{\underline{c}}), \underline{c}(n_{\underline{c}})^\infty]$. This motivates us to introduce the intervals $\mathcal{I}_{\underline{c}}$ via the notion of minimal prefixes.

Definition 5.1 *For a sequence $\underline{c}$ we define the interval $\mathcal{I}_{\underline{c}}$ as the set*

$$\mathcal{I}_{\underline{c}} \; = \; \{\, \underline{x} \in \Sigma_2 : \underline{c}(n_{\underline{c}}) \leq \underline{x} \leq \underline{c}(n_{\underline{c}})^\infty \,\}.$$

The definition of the interval $\mathcal{I}_{\underline{c}}$ is based on a sequence $\underline{c}$, this is however a valid definition as the next theorem shows that the definition is actually independent of the choice of the representative $\underline{c}$. That is, we could have started with any sequences $\underline{d}$ in the interval $\mathcal{I}_{\underline{c}}$ and, via finding its minimal prefix, still would have ended up with same interval.

Theorem 5.2 (Independence) *For any $\underline{d} \in \mathcal{I}_{\underline{c}}$ we have $\mathcal{I}_{\underline{d}} = \mathcal{I}_{\underline{c}}$.*

Proof: Let $\underline{d} \in \mathcal{I}_{\underline{d}} \cap \mathcal{I}_{\underline{c}}$. If $n_{\underline{d}} < n_{\underline{c}}$ then by Lemma 3.18 it would follow that

$$\underline{d} \ \leq \ \underline{d}(n_{\underline{d}})^{\infty} \ = \ \underline{c}(n_{\underline{d}})^{\infty} \ < \ \underline{c}(n_{\underline{c}}),$$

which contradicts that $\underline{d} \in \mathcal{I}_{\underline{c}}$. If we assume $n_{\underline{d}} > n_{\underline{c}}$ then, again by Lemma 3.18,

$$\underline{c}(n_{\underline{c}})^{\infty} \ = \ \underline{d}(n_{\underline{c}})^{\infty} \ < \ \underline{d}(n_{\underline{d}}) \ \leq \ \underline{d}\,,$$

a contradiction and the theorem follows. □

Lemma 5.3 *The dynamical system $\sigma : \mathcal{S}_{\underline{c}} \to \mathcal{S}_{\underline{c}}$ is topological mixing.*

Proof: Let $U \subset \mathcal{S}_{\underline{c}}$ be the cylinder-set defined by $[u_1 u_2 \ldots u_k]$ and let similarly $V \subset \mathcal{S}_{\underline{c}}$ be the cylinder-set defined by $[v_1 v_2 \ldots v_j]$. By choosing the number N sufficiently large the cylinder set $[u_1 u_2 \ldots u_k 1^N v_1 v_2 \ldots v_j]$ is a subset of U. This shows that the intersection $\sigma^n(U) \cap V$ is non-empty for all $n \geq N$. □

We may now return to the postponed proof of the irreducibility of the transition matrices $A_{\underline{c}}$.

Proof of Lemma 4.8: This is an immediate consequences of Lemma 5.3, as each node corresponds to a finite sequence and the mixing property says that we may reach any pattern. That is, for each pair of nodes there exist an N such that there is a path between them of length n for all $n > N$. Among all pairs choose the largest,

N_{max}. Then for any node there is a path to any other node of length n for all $n > N_{max}$, i.e. $A_{\underline{c}}^n > 0$. □

The proof of the second half of the next theorem can be found in Pesin's book [19] on dimension theory. The theorem gives a link between the topological entropy and the Hausdorff dimension via transition matrices for sub-shifts of finite type. The first half is just a restatement of what's obtained in (4.5).

Theorem 5.4 *Let $A_{\underline{c}}$ be a transition matrix with the spectral radius $\rho(A_{\underline{c}})$. Then*

(i). $h_{\text{top}}(\mathcal{S}_{\underline{c}}) = \log \rho(A_{\underline{c}})$;

(ii). $\dim_H(\mathcal{S}_{\underline{c}}) = \frac{\log \rho(A_{\underline{c}})}{\log 2}$.

Example 5.5 The transition matrix A_{01} related to $\mathcal{S}_{\underline{c}}$ for $\underline{c} = 01 \leftrightarrow c = \frac{1}{4}$ is the matrix given in Example 4.1. A_{01} has the eigenvalues $\frac{1}{2}(1+\sqrt{5})$ and $\frac{1}{2}(1-\sqrt{5})$. By the above theorem it follows that the Hausdorff dimension of $\mathcal{S}_{01}$ is

$$\dim_H \mathcal{S}_{01} = \frac{\log\left(1+\sqrt{5}\right)}{\log 2} - 1,$$

or numerically, $\dim_H \mathcal{S}_{01} = 0.694$. □

Our next theorem shows that the intervals $\mathcal{I}_{\underline{c}}$ are maximal intervals, in the sense of being the largest interval of c where $\mathcal{S}_c$ has constant dimension.

Theorem 5.6 (Maximality) *The interval $\mathcal{I}_{\underline{c}}$ is the largest interval on which ϕ is constant.*

Proof: It is clear that the dimension of the set $\mathcal{S}_{\underline{d}}$ is the same for all $\underline{d} \in \mathcal{I}_{\underline{c}}$, this as they all give the same set.

Let $\underline{a_n}$ and $\underline{b_n}$ be the sequences from (3.13). The set $\mathcal{S}_{\underline{a_n}}$ is a proper superset of $\mathcal{S}_{\underline{c}}$ for any positive integer n, since the sequence $\underline{a_n}^{\infty}$ is an element in $\mathcal{S}_{\underline{a_n}}$ but not in $\mathcal{S}_{\underline{c}}$. For n large enough the transition matrix $A_{\underline{a_n}}$ is by Lemma 3.23 and Lemma 4.8 irreducible. We can, by recoding the transition matrix $A_{\underline{c}}$, obtain another transition matrix B for the set $\mathcal{S}_{\underline{c}}$, which is of the same size as $A_{\underline{a_n}}$. We have $A_{\underline{a_n}} \geq B$, where the inequality is strict in at least one entry. By Perron-Frobenius Theorem 4.10 the matrix $A_{\underline{a_n}}$ then have an eigenvalue strictly larger than B and therefore by Theorem 5.4 the dimension increases to the left of $\mathcal{I}_{\underline{c}}$.

Similarly we see that, for any n, the set $\mathcal{S}_{\underline{b_n}}$ is a proper subset of $\mathcal{S}_{\underline{c}}$, as the set doesn't includes the element $\overline{\underline{c}(n_{\underline{c}})}^{\infty}$. We can recode the transition matrix $A_{\underline{c}}$ into a transition matrix C with a size larger than the size of $A_{\underline{b_n}}$, but still irreducible. Next, recode, $A_{\underline{b_n}}$ into the matrix D of the same size as C. This gives the strict inequality for the matrices $C \geq D$, and then also the a strictly larger eigenvalue for C. Hence the decrease of the dimension follows. □

The remaining part is the continuity. To prove it we need the following counting lemma, which gives us estimates of the growth-rate of the number of allowed words. Again, recall the definition of the sequences $\underline{a_n}$ and $\underline{b_n}$ from (3.13).

Lemma 5.7 *(i). There is a constant C such that for all $n \geq 1$*

$$\left|\mathcal{S}_{\underline{a_n}}(n)\right| \;\leq\; C\left|\mathcal{S}_{\underline{c}}(n)\right|.$$

(ii). Given n large, there is a constant C such that for all $k \geq 1$ we have

$$\left|\mathcal{S}_{\underline{b_n}}(k)\right| \;\geq\; C\left|\mathcal{S}_{\underline{c}}(k)\right|\left(1-\frac{1}{n}\right)^{k}.$$

Proof: (i). A sequence $\underline{d}$ in $\mathcal{S}_{\underline{e_n}}(n) \setminus \mathcal{S}_{\underline{c}}(n)$, is a sequence that must start with prefix of a sequence in $\mathcal{S}_{\underline{c}}(n)$ and must end with a prefix of $\underline{a_n}$, as this is the only possibility to get $\underline{d}$ smaller than $\underline{c}$ while shifting. That is,

$$\underline{d} \;=\; \underline{e}(i)\,\underline{a_n}(n-i).$$

with $\underline{e} \in \mathcal{S}_{\underline{c}}(n)$ and $0 \leq i < n$. By what was noticed in (4.4) we have the estimate of the size of $\mathcal{S}_{\underline{a_n}}(n)$ to be

$$\left|\mathcal{S}_{\underline{a_n}}(n)\right| \;\leq\; \sum_{i=1}^{n} \left|\mathcal{S}_{\underline{c}}(i)\right| \;\leq\; k_1 \sum_{i=1}^{n} \lambda^i \;\leq\; k_2 \lambda^n \;\leq\; C\left|\mathcal{S}_{\underline{c}}(n)\right| \tag{5.1}$$

for some constants k_1 and k_2, which is the estimate we sought.

(ii). Similarly, a sequences $\underline{d}$ in the set $\mathcal{S}_{\underline{c}}(k) \setminus \mathcal{S}_{\underline{b_n}}(k)$ must contain, at least once, the pattern $\underline{c}(n_{\underline{c}})^n 0$. The number of such sequences that contains the pattern precisely q times, is bounded by

$$\binom{k - qn\, n_{\underline{c}}}{q} \left|\mathcal{S}_{\underline{b_n}}(k - qn\, n_{\underline{c}})\right|,$$

this, by looking at the number of places the pattern $\underline{c}(n_{\underline{c}})^n 0$ can be placed in. Adopting the notation from (4.4), for the number of sequences, we have by summing up for an n large enough

$$\begin{aligned}
C\;\left|\mathcal{S}_{\underline{c}}(k)\right| \;&\leq\; \lambda_{\underline{b_n}}^{k} + \binom{k - n\, n_{\underline{c}}}{1} \lambda_{\underline{b_n}}^{k-n\, n_{\underline{c}}} + \binom{k - 2n\, n_{\underline{c}}}{2} \lambda_{\underline{b_n}}^{k-2n\, n_{\underline{c}}} + \ldots \\
&\leq\; \lambda_{\underline{b_n}}^{k} \left(1 + \binom{k}{1} \frac{1}{\lambda_{\underline{b_n}}^{n\, n_{\underline{c}}}} + \binom{k}{2} \frac{1}{\lambda_{\underline{b_n}}^{2n\, n_{\underline{c}}}} + \ldots \right) \\
&\leq\; \lambda_{\underline{b_n}}^{k} \left(1 + \frac{1}{\lambda_{\underline{b_n}}^{n\, n_{\underline{c}}}} \right)^{k} \\
&\leq\; \lambda_{\underline{b_n}}^{k} \left(1 + \frac{1}{n} \right)^{k},
\end{aligned}$$

the desired result which concludes the lemma. □

By the help of this counting lemma the final theorem of this section now follows quite easily.

Theorem 5.8 (Continuity) *The map ϕ is continuous.*

Proof: We prove the theorem, by proving the continuity of the topological entropy, and then the result follows from Theorem 5.4. It is clear that for any sequence $\underline{s}$ the estimate $\left|\mathcal{S}_{\underline{s}}(qn)\right| \leq \left|\mathcal{S}_{\underline{s}}(n)\right|^q$ holds. Hence by Lemma 5.7 it follows that

$$\begin{aligned} h_{\text{top}}(\mathcal{S}_{\underline{c}}) &\leq \lim_{n\to\infty} h_{\text{top}}(\mathcal{S}_{\underline{a_n}}) \\ &= \lim_{n\to\infty}\lim_{k\to\infty} \frac{1}{k}\log\left|\mathcal{S}_{\underline{a_n}}(k)\right| \\ &= \lim_{n\to\infty}\lim_{q\to\infty} \frac{1}{qn}\log\left|\mathcal{S}_{\underline{a_n}}(qn)\right| \\ &\leq \lim_{n\to\infty}\lim_{q\to\infty} \frac{1}{qn}\log\left|\mathcal{S}_{\underline{a_n}}(n)\right|^q \\ &\leq \lim_{n\to\infty} \frac{1}{n}\log C\left|\mathcal{S}_{\underline{c}}(n)\right| \\ &= h_{\text{top}}(\mathcal{S}_{\underline{c}}), \end{aligned}$$

which shows the left-continuity of the entropy in the left endpoint of the interval $\mathcal{I}_{\underline{c}}$. The right-continuity follows trivially as the entropy is constant in a neighbourhood to the right of this point. In the same way the left-continuity in the right endpoint of $\mathcal{I}_{\underline{c}}$ is also clear. Again by Lemma 5.7 we have

$$\begin{aligned} h_{\text{top}}(\mathcal{S}_{\underline{c}}) &\geq \lim_{n\to\infty} h_{\text{top}}(\mathcal{S}_{\underline{b_n}}) \\ &= \lim_{n\to\infty}\lim_{k\to\infty} \frac{1}{k}\log\left|\mathcal{S}_{\underline{b_n}}(k)\right| \\ &\geq \lim_{n\to\infty}\lim_{k\to\infty} \frac{1}{k}\log\left(C\left|\mathcal{S}_{\underline{c}}(k)\right|\left(1-\frac{1}{n}\right)^k\right) \\ &= h_{\text{top}}(\mathcal{S}_{\underline{c}}) + \lim_{n\to\infty} \log\left(1-\frac{1}{n}\right) \\ &= h_{\text{top}}(\mathcal{S}_{\underline{c}}), \end{aligned}$$

and the right-continuity in the right endpoints follows and concludes the theorem. $\square$

6. Numerics

For some values of c we can quite easily calculate the dimension of the non-covered set $\mathcal{S}_c$ by considering the transition matrix it gives rise to. Another way is to make a more combinatorial approach, to find the growth rate of the number of allowed sequences and then find the dimension via Theorem 5.4. We show how this can be done for two particular types of sequences.

Proposition 6.1 *(i). Let* $F_n = F_n(\underline{c}) = \left|\mathcal{S}_{\underline{c}}(n)\right|$. *For* $\underline{c} = 0^k 1$ *the number of sequences of length n is given by the recursion*

$$\begin{cases} F_n &= F_{n-1} + F_{n-2} + \ldots + F_{n-k-1}, \\ F_{k+1} &= 2^{k+1} - 1, \\ F_i &= 2^i, \quad 1 \le i \le k. \end{cases} \tag{6.1}$$

(ii). For $\underline{c} = 01^k$ *the number of sequences of length n is given by the recursion*

$$\begin{cases} F_n &= F_{n-1} + F_{n-k-1}, \\ F_i &= i + 1, \quad 1 \le i \le k+1. \end{cases} \tag{6.2}$$

Proof: (i). Denote by $F_n^{\underline{s}}$ the number of sequences of length n that ends with the sequence $\underline{s}$. Thus we have the identity

$$F_n^{0\underline{s}} + F_n^{1\underline{s}} = F_n^{\underline{s}}. \tag{6.3}$$

For the number of sequences of length n we consider the sequence of length k that ends the sequence of length $n-1$. This gives

$$\begin{aligned} F_n &= F_{n-1}^{00\ldots0} + 2F_{n-1}^{00\ldots01} + 2F_{n-1}^{00\ldots10} + \ldots + 2F_{n-1}^{11\ldots1} \\ &= F_{n-1} + F_{n-1}^{00\ldots01} + F_{n-1}^{00\ldots10} + \ldots + F_{n-1}^{11\ldots1} \end{aligned}$$

and repeated use of (6.3) gives

$$F_n = F_{n-1} + F_{n-1}^{1} + F_{n-1}^{10} + \ldots + F_{n-1}^{10\ldots0},$$

which is the first part of (6.1). The initial values follows by a straight forward calculation.

(ii). By considering the ending sequences of the sequences of length $n-1$, as in the above proof, we see that in this case the only ending sequences that can give more than one continuation possibility is $\underline{s} = 111\ldots1$, which gives the term F_{n-k-1} in (6.2). □

Example 6.2 For $\underline{c} = 01$ we have by Proposition 6.1 that the growth-rate of the size of $\mathcal{S}_{01}(n)$ is given by the recursion

$$|\mathcal{S}_{01}(n)| = |\mathcal{S}_{01}(n-1)| + |\mathcal{S}_{01}(n-2)|\,,$$

which simplifies to

$$|\mathcal{S}_{01}(n)| = \frac{1}{\sqrt{5}}\frac{\sqrt{5}+1}{\sqrt{5}-1}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\frac{1-\sqrt{5}}{1+\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

and we obtain by help of Theorem 5.4,

$$\dim_H \mathcal{S}_{01} = \frac{\log\left(1+\sqrt{5}\right)}{\log 2} - 1$$

which can be compared Example 5.5. □

An immediate consequences of the proposition is the following corollary.

Corollary 6.3 *The Hausdorff dimension of $\mathcal{S}_c$ is zero if and only if $c \geq \frac{1}{2}$.*

Proof: In Example 3.1 we saw that $\mathcal{S}_{\underline{c}}$ only contains one element for $\underline{c} = 1$, which show that $\dim_H \mathcal{S}_{\underline{c}} = 0$. The proposition above says that for $\underline{c_k} = 01^k$ the dimension of $\mathcal{S}_{\underline{c_k}}$ is non zero for all k, and the result follows. □

Following the scheme in Example 5.5 of finding the dimension via the transition matrix gives a quite straight forward way to numerically find with a computer a rough picture of the graph of the map $c \mapsto \dim_H \mathcal{S}_c$. The pseudocode for the algorithm looks like:

$L \quad \leftarrow \quad$ list of all minimal sequences of length ≤ 8
for all $\underline{c} \in L$ **do**
  $x_1 \leftarrow$ valueof($\underline{c}$)
  $x_2 \leftarrow$ valueof($\underline{c}^\infty$)
  $n \leftarrow$ lengthof($\underline{c}$)
  $S \leftarrow$ set of all sequences $\underline{x}$ of length n such that $\sigma^k(\,\underline{x}\,) \geq \underline{c}$ for $k = 1, 2, \ldots, 2n$
  $A \leftarrow$ the $2^{n-1} \times 2^{n-1}$ zero-one matrix with $(A)_{ij} = 1$ if $\underline{s}(1, n-1) = i$ and $\underline{s}(2, n) = j$ with $\underline{s} \in S$
  $y \leftarrow$ log (maxeigenvalue(A))/log(2)
  drawline(x_1, y, x_2, y)
endfor

If we would include in the program above minimal sequences of length longer than 8 then it will give a finer partition of the interval $[0, 1]$ but the refinement would be to small to be visible. Secondly it will drastically increase the size of the transition matrix, which grows exponentially, and thereby increase the runtime complexity of the program.

The result of running an implementation of the program is given in Figure 1. Notice the significant difference to Kahane's linear result (1.4) for the covering with randomly placed intervals.

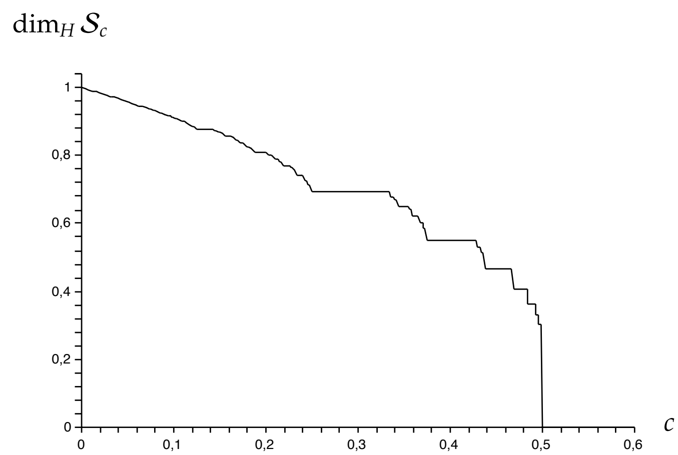


Figure 1: The graph of $\phi(c) = \dim_H \mathcal{S}_c$.

7. Two-sided dyadic covering model

In the one-sided dyadic covering model defined in section 1.3 and discussed thereafter, the intervals $\{I_n\}$ where attached to the system $\{X_n\}$ in their left endpoint. The question dealt within this section is to see what will happen if we attach $\{X_n\}$ to the mid-point of the intervals instead, that is, we create a two-sided model.

Correspondingly to previous sections we will again consider the set of points never covered in this model, we denote the set of such points by $\mathcal{S}^*_c$. As will be seen there are some values of c where the dimension of this set coincides with the dimension of $\mathcal{S}_c$, but that this does not hold in general.

7.1 Reformulation into sequences

We will attack the problem as done in previous sections, we will study the problem transformed into it sequence version. To do this, we see that we may rewrite (1.14) in terms of sequences, and then obtain,

$$\mathcal{S}^*_{\underline{c}} = \{\, \underline{x} \in \Sigma_2 : \left\|\sigma^n(\underline{x})\right\| \geq \sigma^{-1}(\underline{c}),\ \text{for all } n \geq 1 \,\} \qquad (7.1)$$

where we have made the assumption that $\sigma^{-1}(\underline{c}) = 0\underline{c}$, since if we would have inserted a 1 at the front then we would have obtained a lower bound larger than $\frac{1}{2}$, giving the empty-set. We shall hereafter generally assume $\sigma^{-1}(\underline{d}) = 0\underline{d}$ for any sequence $\underline{d}$. By the norm we shall mean

$$\|\underline{x}\| \geq \underline{c} \quad \Leftrightarrow \quad \underline{c}' \geq \underline{x} \geq \underline{c},$$

with the extra notation adopted from the proof of Theorem 3.13, that is, $\underline{c}'$ is the sequence $\underline{c}$ having changed zeros to ones and vice versa.

7.2 Characterisation of the two-sided model

Let us start by giving two small examples enlightening some of the properties of the set $\mathcal{S}^*_{\underline{c}}$ and pointing out some differences in its structure compared to the set $\mathcal{S}_{\underline{c}}$.

Example 7.1 As in the one-sided case we can for some sequences $\underline{c}$, easily describe the set $\mathcal{S}^*_{\underline{c}}$. The two trivial examples are when $\underline{c}$ is the sequence 0 or 1. The first one gives the whole set Σ_2 while the second gives the set $\{(01)^\infty, (10)^\infty\}$. By letting $\underline{c_k} = 0^k1$ we have by Example 3.1 and the identity (1.15),

$$\mathcal{S}^*_{\underline{c_k}} = \left\{ \underline{x} \in \Sigma_2 : \begin{array}{l} \underline{x} \text{ does not contain } k+2 \text{ consecutive} \\ \text{zeros nor ones} \end{array} \right\}$$

The extra +1 compared to Example 3.1 comes from the right shifting of $\underline{c}$ in (7.1). Note in particular that an element $\underline{x}$ of $\mathcal{S}^*_{\underline{c}}$ for some sequence $\underline{c}$ may not contain a block of consecutive zeros nor ones which is one longer than the first block of zeros of $\underline{c}$. It is also worth noting that $\underline{x}$ is an element of $\mathcal{S}^*_{\underline{c}}$ if and only if $\underline{x}' \in \mathcal{S}^*_{\underline{c}}$, which is a property clearly not valid in the one-sided model. □

Example 7.2 Let $\underline{c} = 00111$ and let $\underline{x}$ be an element of $\mathcal{S}^*_{\underline{c}}$. Assume that $\underline{x}$ at position n contains the pattern 000.

$$\underbrace{111001}_{\sigma^{-1}(\underline{c})'} \geq \underbrace{000\ldots}_{\sigma^{n}(\underline{x})} \geq \underbrace{000111}_{\sigma^{-1}(\underline{c})}$$

By what is seen just above, the pattern 000 must be followed by 111, otherwise $\underline{x}$ will fall below the lower bound when continuing the shifting. If so, and by shifting three additional steps we then reach

$$\underbrace{111001}_{\sigma^{-1}(\underline{c})'} \geq \underbrace{111\ldots}_{\sigma^{n+3}(\underline{x})} \geq \underbrace{000111}_{\sigma^{-1}(\underline{c})}$$

and we see that $\underline{x}$ may continue by, either 001 or 000. If it will be the case of 001 $\underline{x}$ must be continued by 0010^∞, otherwise it will be to large. But then by shifting another three steps it will fall below. Hence the only continuation possibility is 000, which puts us back in the case we started from. To conclude, if $\underline{x}$ contains the patterns 000 or 111 then $\underline{x}$ must end with $(000111)^\infty$. $\square$

The previous example indicates a major difference between the one-sided and the two-sided model. From the example it follows that the mixing property of the one-sided system does not hold in the general two-sided case, more explicitly that is

$$\sigma : \mathcal{S}^*_{\underline{c}} \to \mathcal{S}^*_{\underline{c}}$$

is not mixing for all $\underline{c}$. To see how the property fails to hold follows by considering the two cylinders $U = [000]$ and $V = [111]$ as subsets of $\mathcal{S}^*_{\underline{c}}$ for $\underline{c} = 00111$. These are, as seen in the above example, actually trivial cylinders as they are both singleton sets, $U = \{(000111)^\infty\}$ and $V = \{(111000)^\infty\}$. Clearly there is an n such that

$$\sigma^n(U) \cap V \neq \emptyset,$$

but for all such n we have that it will collapse into the empty set for shifting one additional time, which shows the failure of the mixing property.

Next we will prove one of the main observations of this section.

Theorem 7.3 *There are $c \in (0, 1)$ such that $\phi(c) \neq \phi^*(c)$.*

Proof: Let $\underline{d_k} = 001^k$ for $k \geq 4$ and let $\underline{x}$ be an element of $\mathcal{S}^*_{\underline{d_4}}$ containing the pattern 000 at position n. As in Example 7.2 we have

$$\underbrace{1110001}_{\sigma^{-1}(\underline{d_4})} \;\geq\; \underbrace{000\ldots}_{\sigma^n(\underline{x})} \;\geq\; \underbrace{0001111}_{\sigma^{-1}(\underline{d_4})}$$

and we see that $\underline{x}$ must continue with 1111, but then by shifting three additional steps it will be to large. Hence we may conclude that an element of $\mathcal{S}^*_{\underline{c}}$ may not contain three consecutive zeros nor ones. But likewise we could have considered $\underline{d_5}$ and reached the same conclusion. This gives

$$\mathcal{S}^*_{\underline{d_4}} = \mathcal{S}^*_{\underline{d_5}} = \ldots = \mathcal{S}^*_{\underline{d_k}} \tag{7.2}$$

for $k \geq 4$. The sequences $\underline{d_k}$ are all minimal, hence the dimension of $\mathcal{S}_{\underline{d_k}}$ decreases as k increases. This gives that there is at most one k such that

$$\dim_H \mathcal{S}_{\underline{d_k}} = \dim_H \mathcal{S}^*_{\underline{d_k}}$$

which concludes the proof. □

A consequences of (7.2) is that the notion of minimal sequences and the intervals $\mathcal{I}_{\underline{c}}$ connected to them in the one-sided model does not hold in the same sense in the two-sided case, that is, being the largest intervals with constant dimension. But however, there are intervals in the two-sided case, like in the one-sided model, where the dimension of $\mathcal{S}^*_c$ remains unchanged. We shall shortly see that in the two-sided case the dimension remains the same on some intervals, even though we obtain different shifts. But first some result on intervals where the shift is unchanged.

Proposition 7.4 *The set $\mathcal{S}^*_{\underline{c}}$ is unchanged for*

$$\underline{c}(n_{\underline{c}}) \;\leq\; \underline{c} \;\leq\; \sigma\Big((\sigma^{-1}(\,\underline{c}(n_{\underline{c}})))^{\infty}\Big). \tag{7.3}$$

Proof: Using that $\mathcal{S}_{\underline{c}}$ remains unchanged for $\underline{c}$ on the interval $\mathcal{I}_{\underline{c}}$ and the identity (1.15) we may derive for a finite $\underline{c}$

$$\mathcal{S}^*_{\sigma(\underline{c}^{\infty})} = \mathcal{S}_{\underline{c}^{\infty}} \cap \Big(\mathcal{S}_{\underline{c}^{\infty}}\Big)' = \mathcal{S}_{\underline{c}} \cap \Big(\mathcal{S}_{\underline{c}}\Big)' = \mathcal{S}^*_{\sigma(\underline{c})}.$$

In particular this holds for $\underline{c} = \underline{c}(n_{\underline{c}})$. Using the above identity for $\underline{d} = \sigma^{-1}(\,\underline{c}(n_{\underline{c}})\,)$ gives the desired result. □

It is quite clear that the interval in the above proposition is a sub-interval of the interval $\mathcal{I}_c$. Notice that it does not say that we may identify the intervals of constant dimension via minimal sequences as for the one-sided model, where each minimal sequence correspond to a unique interval. The interval above might very well fall into some larger interval giving constant dimension. We shall later-on see that i the case.

Example 7.5 For $\underline{c} = 01$ the right-handside of (7.3) turns into

$$\sigma\left((\sigma^{-1}(01))^{\infty}\right) = \sigma((001)^{\infty}) = 01(001)^{\infty},$$

which then says, via a straight forward calculation of summing the right-hand side up, that $\mathcal{S}_c^*$ has constant dimension on, at least, the interval $[\frac{1}{4}, \frac{2}{7}]$. More generally, we have by examine sequences of the form $\underline{c_k} = 0^k 1$ that

$$\underline{c_k} = 0^k 1 \leftrightarrow \frac{1}{2^{k+1}},$$

and

$$\sigma\left((\sigma^{-1}(\underline{c_k}))^{\infty}\right) = 0^k 1(0^{k+1}1)^{\infty} \leftrightarrow \frac{1}{2^{k+1}}\left(1 + \sum_{i=1}^{\infty} \frac{1}{2^{(k+2)i}}\right) = \frac{2}{2^{k+2}-1}.$$

Hence, the map ϕ^* is constant on the intervals

$$\left[\frac{1}{2^{k+1}}, \frac{2}{2^{k+2}-1}\right]$$

for k larger than or equal to 0. □

In Theorem 7.3 we saw that there exists $c \in (0,1)$ such that the sets $\mathcal{S}_c$ and $\mathcal{S}_c^*$ have different dimension. The opposite is also true, that is, there are values of c where their dimension coincide. To see this we need the following proposition, which goes in the spirit of Proposition 6.1.

Proposition 7.6 *Let $T_n = T_n(\underline{c}) = \left|\mathcal{S}^*_{\underline{c}}(n)\right|$. For $\underline{c} = 0^k 1$ the number of sequences of length n is given by the recursion*

$$\begin{cases} T_n &= T_{n-1} + T_{n-2} + \ldots + T_{n-k-1}, \\ T_{k+1} &= 2^{k+2} - 2, \\ T_i &= 2^i, \quad 1 \leq i \leq k. \end{cases} \tag{7.4}$$

Proof: For the number of sequences of length n we consider the sequence of length $k+1$ that ends the sequence of length $n-1$. This gives

$$\begin{aligned} T_n &= T^{00\ldots0}_{n-1} + 2T^{00\ldots01}_{n-1} + 2T^{00\ldots10}_{n-1} + \ldots + 2T^{11\ldots10}_{n-1} + T^{11\ldots11}_{n-1} \\ &= T_{n-1} + T^{00\ldots01}_{n-1} + T^{00\ldots10}_{n-1} + \ldots + T^{11\ldots10}_{n-1} \end{aligned}$$

and repeated use of (6.3) gives

$$\begin{aligned} T_n &= T_{n-1} + \left(T^{01}_{n-1} + T^{10}_{n-1}\right) + \left(T^{001}_{n-1} + T^{110}_{n-1}\right) \ldots + \left(T^{00\ldots01}_{n-1} + T^{11\ldots10}_{n-1}\right) \\ &= T_{n-1} + \left(T^{0}_{n-2} + T^{1}_{n-2}\right) + \left(T^{0}_{n-3} + T^{1}_{n-3}\right) \ldots + \left(T^{0}_{n-k-1} + T^{1}_{n-k-1}\right), \end{aligned}$$

which is the first part of (7.4). The initial values follows by a straight forward calculation. □

The positive answer to the question if equality (1.16) holds now follows quite directly.

Theorem 7.7 *The dimension of $\mathcal{S}_c$ and $\mathcal{S}^*_c$ coincide for c in the set*

$$C = \bigcup_{k=1}^{\infty} \left[\frac{1}{2^{k+1}}, \frac{2}{2^{k+2} - 1}\right] \cup \left[\frac{1}{2}, 1\right).$$

Proof: From the above proposition we conclude, via Theorem 5.4 that the equality (1.16) holds for $\underline{c_k} = 0^k 1$, $k \geq 1$, and therefore, from Example 7.5 the result follows by also noticing that both sets have dimension zero on $[\frac{1}{2}, 1)$. □

Next, let us turn to the question whether the interval in (7.3) is maximal in the sense of being the largest interval where ϕ^* is constant. We shall see that in some cases we may extend the interval in (7.3) leftward. To do this, we shall consider how the set $\mathcal{S}^*_{\underline{c}}$ increases when decreasing $\underline{c}$. If we decrease $\underline{c}$ carefully it will increase $\mathcal{S}^*_{\underline{c}}$ but not fast enough to affect the dimension.

In Example 7.2 the elements of $\mathcal{S}^*_{\underline{c}}$, for $\underline{c} = 00111$ must all end with $(000111)^\infty$ if they contain either of the patterns 000 or 111. By Example 7.1 we see that an element of $\mathcal{S}^*_{\underline{c}}$ is either an element of $\mathcal{S}^*_{\underline{d}}$, for $\underline{d} = 01$ or starting with a prefix of an element of $\mathcal{S}^*_{\underline{d}}$ followed by $(000111)^\infty$ or $(111000)^\infty$. Hence we have the estimate

$$\left|\mathcal{S}^*_{\underline{c}}(n)\right| \le \sum_{k=1}^{n} \left|\mathcal{S}^*_{\underline{d}}(k)\right| \le C\left|\mathcal{S}^*_{\underline{d}}(n)\right|$$

as the size of $\mathcal{S}^*_{\underline{d}}(n)$ grows like λ^n, for some $1 \le \lambda \le 2$. By Theorem 5.4 we see that the dimension of $\mathcal{S}^*_{\underline{c}}$ is constant when $00111 \le \underline{c} \le 01$. Comparing with (7.3) we see that this give the extension

$$\dim_H \mathcal{S}^*_{00111} = \dim_H \mathcal{S}^*_{01} = \dim_H \mathcal{S}^*_{01(001)^\infty}$$

that is, the dimension is constant on the interval

$$\left[\frac{7}{32}, \frac{2}{7}\right] = [0.2188, 0.2857],$$

and therefore the interval (7.3) is not maximal.

We end the section by giving a numerical visualisation of the graph of the map $\phi^*(c) = \dim_H \mathcal{S}^*_c$. Via the same numerical methods as in Section 6. we obtain an approximative picture of the graph of the map ϕ^*, see Figure 2.

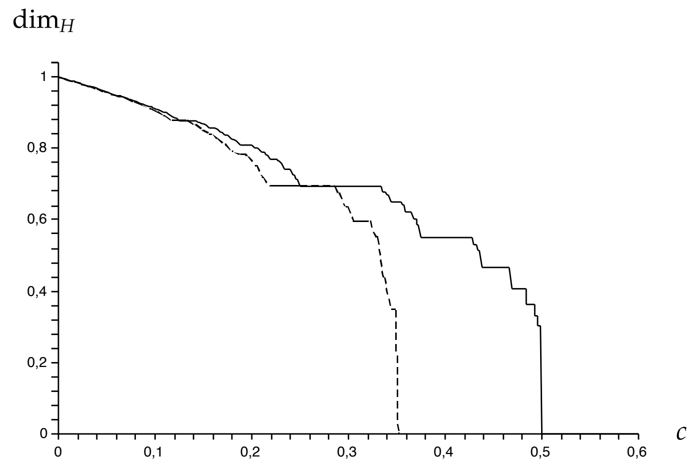


Figure 2: The graphs of $\phi(c) = \dim_H \mathcal{S}_c$ and $\phi^*(c) = \dim_H \mathcal{S}_c^*$ (dotted).

7.3 Substitutions and combinatorics

In this section shall show that in one special case, the case of having dimension equal to zero, we are able to find the left endpoint of an interval where the dimension of $\mathcal{S}_c^*$ is constant. As shall be seen the endpoint is in relation to the classical Thue-Morse sequence, a is a sequence that appears in many areas such as combinatorics on words and the study of substitutions. The history around this sequence goes back to the work of Thue, who when working on overlap-free sequences on a finite alphabet, i.e., those sequences that do not contain the pattern $a\underline{x}a\underline{x}a$, introduced this special sequence. For more of this, see [3] for a deeper presentation and further references.

Definition 7.8 *The sequence $\underline{t}$ recursively defined by $t_0 = 0$ and $t_{2n} = t_n, t_{2n+1} = t'_n$, where x' is the opposite symbol to $x \in \{0, 1\}$, is called the Thue-Morse sequence.*

Note the use of the ′-notation for inverses. For a sequence $\underline{x}$ the prime notation is the inverse of the sequences, (as used in previous sections), and for a symbol x_i in $\{0, 1\}$, it just denotes the opposite one. That is, if 1 is considered as a sequence then $1' = 1$, but if it is considered as a symbol, $1' = 0$.

Denote by $s(n)$ the sum of the digits in the binary representation of the integer n. Since we clearly have $s(2n) = s(n)$ and $s(2n+1) = s(n)+1$ for every $n \geq 0$, we easily obtain the equivalent definition:

Definition 7.9 *The Thue-Morse sequence $\underline{t}$ is the sequence defined by $t_n = s(n)$ mod 2 for $n \geq 0$.*

The inverse Thue-Morse sequence $\underline{t}'$ can, besides the usual way, also be obtained if starting with a 1 in Definition 7.8. The Thue-Morse sequence, and its inverse are the sequences starting with,

$$\underline{t} = 0110100110010110100101 10\ldots$$

$$\underline{t}' = 1001011001101001011010 01\ldots$$

Before leaving the presentation of the Thue-Mores sequence, let us give yet an additional way of obtaining it, via *substitutions*. Let μ be a morphism defined by $\mu(0) = 01$ and $\mu(1) = 10$. Then the infinite fixed-point under μ beginning with 0 is the Thue-Morse sequence $\underline{t}$. Clearly we then obtain $\underline{t}'$ as the fix-point of μ if starting with 1.

This is however not the end of the list of ways of obtaining the Thue-Morse sequence. In [3] Allouche and Shallit present a wide range of areas where this special sequence appears and how it gives rise to numerous applications.

We introduce here the map f, which gives a reflected self-concatenation. We shall see later on that this map is connected to the Thue-Morse sequence. The map f is the following;

Definition 7.10 *For a finite sequence $|\underline{e}| = n$, ending with a 1, we define the map f by*

$$f(\,\underline{e}\,) = \underline{e}(n-1)0\left(\underline{e}(n)\right)'. \tag{7.5}$$

Note that f can also be seen as a function from $\mathbb{Q}$ to $\mathbb{R}$. Let us give two short examples showing some basic properties of the above defined map f and to illustrate the mentioned sequence doubling via reflections.

Example 7.11 We have

$$f(01) = (0)0(01)' = 0011,$$

$$f^2(01) = f(0011) = (001)0(0011)' = 00101101$$

and similarly

$$f^3(01) = 0010110011010011.$$

□

From Definition (7.10) we see that, for a sequence $\underline{e}$, $|\underline{e}| = n > 1$, the function f is monotonically decreasing under composition and is bounded from below,

$$\underline{e}(n-1) < f^k(\underline{e}) < f^{k-1}(\underline{e}). \tag{7.6}$$

for all $k \geq 1$. Hence the limit $\lim_{k\to\infty} f^k(\underline{e})$ exists.

Numerically the map f gives the following; let $\underline{e}$ correspond to the dyadic number $\frac{a}{2^n}$. Then when applying f we obtain

$$f(\tfrac{a}{2^n}) = \frac{a}{2^n}\left(1 - \frac{1}{2^n}\right).$$

Generalising this to the infinitely repeated self-compositions gives the identity

$$\lim_{k\to\infty} f^k(\tfrac{a}{2^n}) \;=\; \frac{a}{2^n}\prod_{i=0}^{\infty}\left(1 - \frac{1}{2^{2^i n}}\right), \tag{7.7}$$

which is convergent and by (7.6) bounded away from zero.

Remark: With a theorem of Mahler, or by Roth's theorem, it can be shown that the real number corresponding to the product in (7.7) is transcendental.

Let us for the rest of the section introduce the notation for self-compositions of f for all $k \geq 0$ starting with the sequence 01,

$$\underline{d_k} = f^k(01).$$

In Example 7.11 some of the first $\underline{d_k}'s$ are given. By what's seen above, $\underline{d_k}$ converges when k tends to infinity. Let us denote this limit point by $\underline{d}$,

$$\underline{d} \;=\; \lim_{k\to\infty} f^k(01).$$

It is not hard to calculate the first symbols of $\underline{d}$, as the convergence of $\underline{d_k}$ is quite fast,

$$\underline{d} = 00101100110100101101001100101100\ldots$$

The resemblance to the inverse Thue-Morse sequence is not a coincidence, as the following proposition shows.

Proposition 7.12 *The sequence* $1\underline{d}$ *is the inverse Thue-Morse sequence, i.e.* $\underline{d} = \sigma(\underline{t}')$.

Proof: Let us denote by $\underline{w}$ the sequence $1\underline{d}$ and where we have slightly changed the indices such that we start the enumeration form zero. That is

$$\underline{w} = w_0 w_1 w_2 \ldots = 10010110\ldots.$$

For an integer $q \geq 1$ let n_1 be the largest integer such that $2^{n_1} \leq q$. By the definition of f we have that the symbol at position $q - 2^{n_1}$ in $\underline{w}$ is the opposite to the symbol at position q. Recursively we let n_2 be the largest integer such that $2^{n_2} \leq q - 2^{n_1}$. As above, the symbol at position $q - 2^{n_1} - 2^{n_2}$ is the inverse of the symbol at position $q - 2^{n_1}$. Hence by continuing this way we see that the symbol at position q is fixed by the number of ones in the binary expansion of q, but this is precisely the definition of the inverse Thue-Morse sequence via Definition 7.9. $\square$

The idea we are going to use to prove the main result in this section is that each $\underline{d_k}$ is an extension left-ward of the interval where the dimension of $\mathcal{S}_c^*$ is zero, and then show that the interval can not be extended beyond the point, $\underline{d}$. For this we need the following strong result of shift-estimations of $\underline{t}$.

Theorem 7.13 (Allouche) *Let $\underline{t}$ be the Thue-Morse sequence and let $\underline{a} = \sigma(\underline{t})$. Then for each $n \geq 1$, we have*

$$\underline{a}' < \sigma^n(\underline{a}) < \underline{a}.$$

This theorem falls into the study of combinatorics on words and we refer to the works of Allouche and others [1, 2] for a complete proof of it. From Theorem 7.13 we obtain the following useful corollary,

Corollary 7.14 *For all $k \geq 1$ we have $\underline{d_k}^\infty \in \mathcal{S}^*_{\sigma(\underline{d_k})}$.*

Proof: Theorem 7.13 and Proposition 7.12 tells us, via considering the inverses,

$$\underline{d}' > \sigma^n(\underline{d}) > \underline{d}. \tag{7.8}$$

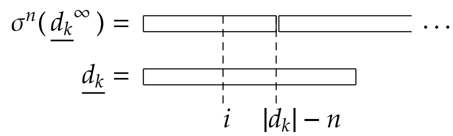
for all $n \geq 1$. We wish to prove that for a fixed k the inequalities

$$\underline{d_k}' \;\geq\; \sigma^n(\underline{d_k}^\infty) \;\geq\; \underline{d_k} \tag{7.9}$$

hold for all $n \geq 1$. Let us start with the right-handside inequality. We assume for contradiction that there exists an $n \leq |\underline{d_k}|$ such that

$$\sigma^n(\underline{d_k}^\infty) \;<\; \underline{d_k} \tag{7.10}$$

and denote the position were the first difference occurs by i. It is clear that if $i < |\underline{d_k}| - n$ we would reach a contradiction to (7.8).



Therefore assume $i \geq |\underline{d_k}| - n$. Let $\underline{e_k}$ be the sequence $\underline{d_k}$ where the ending 1 has been replaced by a zero. From (7.8) it follows that

$$\begin{aligned}
\sigma^n(\underline{d_k})(|\underline{d_k}| - n) &> \sigma^n(\underline{e_k})(|\underline{d_k}| - n) \\
&= \sigma^n(\underline{d})(|\underline{d_k}| - n) \\
&\geq \underline{d}(|\underline{d_k}| - n) \\
&= \underline{d_k}(|\underline{d_k}| - n)
\end{aligned}$$

which contradicts (7.10). Hence the right-hand side inequality of (7.9) follows. For proving the left-hand side inequality in (7.9), notice that by the special property of the $\underline{d_k}$ sequences

$$(\underline{d_k}^\infty)' = \sigma^{|\underline{d_k}|/2}(\underline{d_k}^\infty).$$

Then by the just proved inequality we have

$$\sigma^n((\underline{d_k}^\infty)') \geq \underline{d_k},$$

which holds for $n \geq 1$. The desired result now follows by taking the inverse. $\square$

The identity (7.8) tells us that the sequence $\underline{d}$ is an element of $\mathcal{M}$, (see (3.4)), that is, $\underline{d}$ is an infinitely minimal sequence. Also, note that the above lemma and Lemma 3.19 implies that the $\underline{d_k}'s$ are minimal sequences, as they are not periodic.

We can now give the first part of the result stating how much we can extend left-ward without increasing the dimension.

Lemma 7.15 *For all $k \geq 0$ we have* $\dim_H \mathcal{S}^*_{\sigma(\underline{d_k})} = 0$.

Proof: For $k = 0$ we have $\mathcal{S}^*_1 = \{(01)^\infty, (10)^\infty\}$, so the dimension is zero. Now let

$$\underline{x} \in \mathcal{S}^*_{\sigma(\underline{d_k})} \setminus \mathcal{S}^*_{\sigma(\underline{d_{k-1}})}.$$

Corollary 7.14 says that the set on the right-hand side above is non empty, which justifies the assumption. We have that $\underline{x}$ must contain the pattern $\overline{\underline{d_k}}$ at least once, otherwise it would be an element of $\mathcal{S}^*_{\sigma(\underline{d_{k-1}})}$. Assume that the first time this occurs is at position n. By shifting the sequence $\underline{x}$, $n + |\underline{d_k}|/2$ times we see that the pattern $\overline{\underline{d_k}}$ must be repeated. Hence, as we saw in the previous section, if $\underline{x}$ contains the pattern $\overline{\underline{d_k}}$ then it must end with $\underline{d_k}^\infty$.

This gives that $\underline{x}$ may start with a prefix of a sequence in $\mathcal{S}^*_{\sigma(\underline{d_{k-1}})}$ and then end with an infinite repetition of $\underline{d_k}$, (or its inverse). Hence by using the same idea as in (5.1) we have the estimate of the size

$$\left|\mathcal{S}^*_{\sigma(\underline{d_k})}(n)\right| \leq \sum_{i=1}^{n} \left|\mathcal{S}^*_{\sigma(\underline{d_{k-1}})}(i)\right| \leq C_1 \sum_{i=1}^{n} \lambda^i \leq C_2 \lambda^n \leq C_3 \left|\mathcal{S}^*_{\sigma(\underline{d_{k-1}})}(n)\right|.$$

Hence the extension does not change the entropy. The lemma follows by repeating this argument for all $k \geq 1$ and by applying Theorem 5.4. $\square$

Note that from the proof of the above lemma we see that an element $\underline{x}$ of $\mathcal{S}^*_{\sigma(\underline{d_k})}$ may contain an increasing pattern $\underline{d_{k-1}}\,\underline{d_k}$ but it may not contain the descending one, $\underline{d_k}\,\underline{d_{k-1}}$.

The following theorem says that the left end-point obtained via our method of using the $\underline{d_k}$ sequences, for extending the constant intervals in the above Lemma 7.15, really is the left end-point of this constant interval.

Theorem 7.16 *The Hausdorff dimension of* $\mathcal{S}^*_{\sigma(\underline{c})}$ *is zero if and only if* $\underline{c} \geq \sigma(t')$.

Proof: The previous lemma says that $\dim_H \mathcal{S}^*_{\sigma(\underline{c})} = 0$ for $\underline{c} \geq \sigma(\underline{t}')$. For proving that this lower bound can not be improved we start by defining the sequences

$$\underline{e_k} = \underline{d_k}(2^k + 1)$$

for $k \geq 1$. As seen previously the sequence $\{\underline{d_k}\}$ converges to a sequence $\underline{d}$. By construction we also have the sequence $\underline{d}$ being the limit of the sequences $\underline{e_k}$ when k tends to infinity. Moreover, for all $k \geq 1$, we have the inequalities

$$\underline{e_k} < \underline{e_{k+1}} < \underline{d} < \underline{d_{k+1}} < \underline{d_k}. \tag{7.11}$$

The idea in the proof is to show that, $\mathcal{S}^*_{\sigma(\underline{e_k})}$ contains sequences containing the descending pattern $\underline{d_k}\,\underline{d_{k-1}}$, and then create a subset of $\mathcal{S}^*_{\sigma(\underline{e_k})}$ isomorphic to Σ_2. To prove the existence of sequences containing a descending pattern it is enough to prove that the sequence

$$\underline{x} = \underline{d_k}\left(\underline{d_{k-1}}\right)^{\infty}$$

is an element of $\mathcal{S}^*_{\sigma(\underline{e_k})}$, that is we have to convince ourselves that

$$\underline{e_k}' \;\geq\; \sigma^n(\,\underline{x}\,) \;\geq\; \underline{e_k} \tag{7.12}$$

holds for all $n \geq 1$. It is clear that both $\underline{d_k}^{\infty}$ and $\underline{d_{k-1}}^{\infty}$ are elements of $\mathcal{S}^*_{\sigma(\underline{e_k})}$, by the inclusion indicated by (7.11) and by Corollary 7.14. Hence we have that if (7.12) holds for $n < |\underline{d_k}| = 2^{k+1}$ then it holds for all n, as then the prefix $\underline{d_k}$ of $\underline{x}$ has been shifted out. But by noticing that $\underline{d_{k-1}}(2^k - 1)$ is a prefix of $\underline{d_k}$ it follows that (7.12) holds for

$$n \leq |\underline{d_k}| + \left|\underline{d_{k-1}}(2^k - 1)\right| - |\underline{e_k}| = 2^{k+1} - 2.$$

This leaves one single case needing special treatment, $n = 2^{k+1} - 1$. The validity of (7.12) in this final case follows as $\underline{e_k}$ starts with 00, and therefore $\underline{e_k}'$ begins with 11, while $\sigma^n(x) = 1\underline{d_{k-1}}^{\infty} = 10\ldots$. Hence we conclude that $\underline{x}$ is an element of $\mathcal{S}^*_{\sigma(\underline{e_k})}$.

Define for $k \geq 1$ the sets

$$E_k = \left\{ \underline{d_{k-1}^2}, \underline{d_k} \right\}^{\mathbb{N}}.$$

As (7.12) holds for all $n > 0$ and since we know that the pattern $\underline{d_{k-1}}\,\underline{d_k}$ is also a valid pattern in sequences in $\mathcal{S}^*_{\sigma(\underline{e_k})}$ it follows that E_k is a subset of $\mathcal{S}^*_{\sigma(\underline{e_k})}$.

Define the map $g_k : \Sigma_2 \to E_k$ such that it maps $0 \mapsto \underline{d_{k-1}^2}$ and $1 \mapsto \underline{d_k}$. As $\underline{d_{k-1}^2} \neq \underline{d_k}$ and they are of the same length, 2^{k+1}, it follows that g_k is a bijection. Moreover, for the distance $\delta = 2^{-i}$ where i is position of the first difference, we have the Hölder condition

$$\delta\left(g_k(\underline{u}), g_k(\underline{v})\right) \;\geq\; C\,\delta\left(\underline{u}, \underline{v}\right)^{2^{k+1}}$$

for some constant C. This implies via Proposition 2.2 the estimate of the dimension

$$\begin{aligned} \dim_H \mathcal{S}^*_{\sigma(\underline{e_k})} &\geq \dim_H E_k \\ &\geq \frac{1}{2^{k+1}} \dim_H g_k^{-1}(E_k) \\ &= \frac{1}{2^{k+1}} \dim_H \Sigma_2 \\ &= \frac{1}{2^{k+1}} > 0, \end{aligned}$$

and we may conclude that $\mathcal{S}^*_{\sigma(\underline{e_k})}$ has a non-zero dimension for all $k \geq 1$. □

7.4 Conclusion

To conclude, we emphasise that the two-sided covering model lacks in general the property of mixing, which is central in the one-sided case. But the two models still show some minor similarities, that of being partly constant and numerically they seem to be both Cantor staircases. But any further and deeper similarities are unlikely to find as they give rise to two completely different shifts.

Let us end by reformulating Theorem 7.16 into a more explicit way, it says that the Hausdorff dimension of the set

$$\{\, x \in \mathbb{S} \;:\; \|2^n x\| \geq c, \text{ for all } n \geq 1 \,\}$$

is zero if and only if $c \geq 1-2\tau$, where τ is the Thue-Morse constant, that is $c \geq 0.17509193\ldots$.

8. Further extensions

Our brief investigation of the two-sided covering model in the previous section leaves, among others, a couple of interesting questions unanswered, namely :

1. Does the two-sided covering model share the same fractal properties as the one-sided?

2. Is the set C of c-values the maximal set where the one-sided and two-sided coincide, and more over do we always have $\dim_H \mathcal{S}_c^* \leq \dim_H \mathcal{S}_c$?

3. The Section 7.3 poses the question if all endpoints of constant intervals of the dimension are connected to fix-points under iterations of the map f. What do the right endpoints look like?

4. What will happen if we will attach $\{X_n\}$ to the intervals in a more general position, that is, if we for some $0 \leq d \leq 1$ we let

$$I_n = X_n + \frac{1}{2}\big(-l_n(1-d),\, l_n(1+d)\big)$$

be the covering intervals?

References

[1] Allouche J. P., *Théorie des nombres et automates,* Thèse d'État, Université Bordeaux I, (1983).

[2] Allouche J. P., Currie J., Shallit J., *Extremal infinite overlap-free binary words,* The Electronic Journal of Combinatorics. 5, (1998), #27.

[3] Allouche, J. P., Shallit, J. *The Ubiquitous Prouhet-Thue-Morse Sequence,* In Sequences and Their applications, Proc. SETA'98 (Ed. C. Ding, T. Helleseth, and H. Niederreiter). New York: Springer-Verlag (1999) 1-16.

[4] Billard P., *Séries de Fourier aléatoirement bornée, continues, uniformément convergentes*, Ann. Sci. Ecole. Norm. Sup. 83 (1965), 131-179.

[5] Birkhoff G. D., *Proof of the ergodic theorem,* Proceedings of the Academy of Sciences USA 17, (1931), 656-660.

[6] Domb C., *Covering by Random Intervals and One-Dimensional Continuum Percolation*, J. Stat. Phys. 55 (1989), 441 - 460.

[7] Dvoretzky A., *On covering a circle by random placed arcs*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 199–203.

[8] Falconer K., *Fractal geometry,* (Wiley 1990).

[9] Fan A. H., *How many intervals cover a point in the Dvoretzky covering?*, Israel J. Math. 131 (2002), 157–184.

[10] Fan A., Schmeling J., *Coverings of the circle driven by rotations,*

[11] Kahane J. P., *Some random series of functions,* Camebridge University Press (1985).

[12] Katok A., Hasselblatt B., *Introduction to the Modern Theory of Dynamical System,* Cambridge University Press (1995).

[13] Khintchine A., *Continued Fractions*, P. Noordhoff Ltd (1963).

[14] Kitchens B. P., *Symbolic dynamics, one-sided, two-sided and countable state Markov shifts,* Springer (1998).

[15] Mandelbrot B., *Renewal sets and random utouts,* Z. Wahrscheinlichkeitsheorie verw. Geb. 22 (1972), 145-157.

[16] Mandelbrot B., *On Dvoretzky coverings for the circle,* Z. Wahrscheinlichkeitsheorie verw. Geb. 22 (1972), 158-160.

[17] Parry W, *On the β-expansions of real numbers*, Acta Math. Acad Sci. Hung. 11 (1960), 401-416.

[18] Perron. O., *Zur Theorie der Matrices,* Mathematisches Annalen, 64 (1906).

[19] Pesin Y., *Dimension Theory in Dynamical Systems,* The University of Chicago Press (1997).

[20] Rényi A, *Representation for real numbers and their ergodic properties*, Acta Math. Acad Sci. Hung. 8 (1957), 477-493.

[21] Schmeling J., *Symbolic dynamics for β-shifts and self-normal numbers,* Ergod. Th. & Dynam. Sys. 17 (1997), 675-694.

[22] Shepp L., *Covering the circle with random arcs,* Israel J. Math. 11 (1972), 328–345.

[23] Stevens W. L., *Solution to a Geometrical Problem in Probability,* Ann. Eugenics 9 (1939), 315-320.

[24] Urbanski M., *On Hausdorff dimension of invariant sets for expanding maps of the circle,* Ergod. Th. & Dynam. Sys. 6 (1986), 295-309.