Project work – nonlinear dynamical systems

Instructions: Choose a nonlinear dynamical system that is related to a topic/application/problem that you retain intresting. The system may be time-discrete or time-continuous. In the sequel we consider mainly time-continuous problems but corresponding instructions apply to time-discrete systems. In case you don't have an "own" appealing system, you may choose one of the projects described below. You should work on an autonomous system in 2 or 3 dimensions. Also periodically forced oscillators may serve. The system should include "parameters" in such a way that exploration in parameter space will lead to qualitatively different behaviour. Apart from the specific questions arising on each project, the following steps should always be observed:

1. Find equilibrium points and investigate their stability via linearisation. Establish specifically for which parameter values the stability properties of the fixed point(s) change in nature.

2. Investigate if there exists global stability.

3. Decide, possibly analytically if there exist periodic solutions to the system.

4. Simulate enough orbits so that you can qualitatively describe the phase portrait. Repeat this for different parameter values, typical of each qualitatively different behaviour and even for some bifurcation values.

5. Guess more or less the stable and unstable manifolds of the fixed points (maybe also by simulation. Note that you can do numerical simulations backwards in time as well, if needed).

6. For systems of order larger than 2 chaotic attractors may occur. It is not mandatory to perform theoretical analysis around them (but it is not forbidden!). It would be however nice to display numerical evidence of their eventual existence. Analyise the occurrence of sensitivity to initial conditions.

Write a short presentation of your work including the analytical computations, graphical illustrations and numerical work. Include a comment on the applications motivating the problem and a list of references.

Project list

1. **Solutions of a Non-linear Elliptic Equation in a Ball.** Consider the differential equation

$$\Delta u = u^{-\beta} \ in \ \ B_R = \{x \in R^N : |x| < R\}, u = 1 \ on \ \ \partial B_R.$$

It follows from a theorem of Gidas, Ni and Nirenberg that its solutions are all rotationally symmetric. In polar coordinates the solution u satisfies the second order non-linear, non-autonomous differential equation:

$$u''(r) + \frac{N-1}{r} u'(r) = u(r)^{-\beta}.$$

This can be transformed to an autonomous, quadratic system by introducing the variable $\tau = \log r$, and the functions

$$\xi(\tau) = \frac{ru'(r)}{u(r)},$$

and

$$\eta(\tau) = \frac{r^2}{u(r)^{\beta+1}}.$$

Do this transformation, analyze the resulting 2d autonomous system and draw some conclusions about the original partial differential equation, in particular how the number of solutions depends on R, N (positive integer), and β.

2. Analyze the system

$$\ddot{x} + \mu\dot{x} + \nu x + x^2\dot{x} + x^3 = 0,$$

och studera bifurkationerna vid variation av parametrarna.

3. Analyze the system:

$$\begin{aligned} \dot{x} &= x\left(4(1-x^2) - 3\beta y^2\right) \\ \dot{y} &= y\left(3(1-y^2) + (3\beta - 7)x^2\right) \end{aligned}$$

Check in particular the possibility of homoclinic or heteroclinic connections.

4. The love of Romeo and Juliet is described by the following system of differential equations:

$$\begin{aligned}\dot{x} &= ax + by^3 \\ \dot{y} &= ax + by\end{aligned}$$

The variable x describes Juliet's love for Romeo, and the variable y describes his love for her. Positive values mean love and negative values mean hatred. The parameter a describes how many flowers Juliet gets from Romeo. If $a = 0$, then she gets the same amount as the average woman, if $a > 0$, then she gets more and if $a < 0$, then she gets less. The parameter b describes in the same way how much good food Romeo is getting from Juliet. Analyze this system, and interpret what you find.

5. A certain predator-prey system is given by

$$\begin{aligned}\dot{x} &= x(1 - x - ky) \\ \dot{y} &= -y(1 + y - kx)\end{aligned}$$

The variable x is the number of prey, and the variable y is the number of predators. The parameter k measures how much prey a predator needs to survive. Analyze this system, and interpret what you find. Only positive x and y are interesting for the application, but mathematically, negative values might also have an interest.

6. Here is a system with "discrete time" (map): Let

$$f(z) = z^2 + c,$$

for a complex variable z and a complex number c. Determine fixed points and 2-cycles and find out if they are stable for different values of c. The Mandelbrot and Julia sets are related to this system. Describe roughly what these sets are and write a program to plot some of these.

7. A competition model. Consider two populations, which do not prey on each other but compete with each other for a common food supply. Each species in isolation, grows according to a logistic map law $x' = ax(1 - x)$, $y' = by(1 - y)$ (rescaled coordinates, all constants positive). The presence of another species will reduce the available food supply and hence slow down the growth rate of the other species, as follows (positive constants):

$$\begin{aligned}x' &= ax(1 - x - a_2 y) \\ y' &= by(1 - y - b_2 x).\end{aligned}$$

Analyze this map, and show if depending on the parameters, this can lead to coexistence, extinction of one species, or total extinction.

8. Advertising can be likened to spreading germs. Potential buyers x catch these 'germs' through advertisement and contact with brand name users:

$$x' = k - \alpha xy + \beta y$$
$$y' = \alpha xy - (\beta + \epsilon)y.$$

Note that when a potential buyer becomes a user, x decreases and y increases. Here β is the switching rate to a rival product, and ϵ is a (small) factor measuring migration and mortality.

9. The Brusselator: A model for certain chemical reactions. Analyze the following system and discuss its bifurcations:

$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2 y \\ \dot{y} &= bx - x^2 y, \end{aligned}$$

where x and y are nonnegative (chemical concentrations) and a and b are strictly positive.