LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 9
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1. a) Let X be a normed vector space and $x_n, x_\infty \in X$. Prove that $x_n \to x_\infty$ weakly[1] $\Leftrightarrow f(x_n) \to f(x_\infty)$ for all $f \in X^*$.

b) Show that in $\mathbf{R}^n$ the weak convergence coincides with the strong one.

c) Let X be a Hilbert space. Show that $x_n \to x_\infty$ weakly and $\|x_n\| \to \|x_\infty\|$ then $x_n \to x_\infty$ strongly.

d) Define $e_k = \{0, 0, 0, \ldots, 0, 1, 0, \ldots\}$ where 1 takes the place number k. Study the convergence of e_k in weak and weak* sense (if applied) in the following spaces[2]: c_0, ℓ^1, ℓ^p for $1 < p < +\infty$ and ℓ^∞.

e) Prove that in ℓ^1 the weak convergence coincides with the strong one.

2. Let X be a normed vector space. Prove that the norm is

a) continuous in the strong topology.

b) lower semi-continuous in the weak topology.

3. a) Concretize the alignment principle for $x \in \ell^{1+}$ and $y \in \ell^{\infty+}$, i.e. provide the conditions on y_k and x_k for x, y to be aligned.

b) Define $M = \{x \in \ell^{1+} \colon \sum_{k=0}^{+\infty} x_k = 0\}$ and solve the problem $\inf_{x \in M} \|b - x\|_1$ given $b \in \ell^{1+}$ via duality in Problem 1.

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[1] Definition: $x_n \to x_\infty$ if $\forall V$ neighbourhood of x_∞ $\exists N$: $x_n \in V$ $\forall n \geq N$.

[2] The definition of c_0 can be found in the Exercise 1.