LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 7
2013–10–24 kl 10–12

1. Consider a function $f\colon X \to Y$.

a) Show that exactly one of the following identities for the image and pre-image[1] is wrong and prove the other three.

$$\begin{aligned} f(A\cup B) &= f(A)\cup f(B), & f^{-1}(A\cup B) &= f^{-1}(A)\cup f^{-1}(B),\\ f(A\cap B) &= f(A)\cap f(B) & f^{-1}(A\cap B) &= f^{-1}(A)\cap f^{-1}(B) \end{aligned}$$

b) Let (Y,τ_y) be a topological space. Show that $f^{-1}(\tau_y) = \{f^{-1}(G)\colon G\in\tau_y\}$ is a topology on X (an *induced* topology).

c) Let (X,τ_x) be a topological space. Show that $\{H\subset Y\colon f^{-1}(H)\in\tau_x\}$ is a topology on Y (a *co-induced* topology).

d) Show that the induced topology is the weakest topology and the co-induced topology is the strongest topology such that the function f is continuous.

2. Let (X,τ_x) and (Y,τ_y) be topological spaces such that the function $f\colon X\to Y$ is continuous. Exactly one of the following statements is true. Prove it.

a) The image of any compact set is compact.

b) The pre-image of any compact set is compact.

3. Let (X,τ) be a topological space. Prove that $\mathcal{G}$ is a base of the topology τ if and only if

$$\forall G\in\tau\ \forall x\in G\ \exists G_x\in\mathcal{G}\colon x\in G_x\subset G.$$

4. Let (X,τ) be a topological space. Let K be a compact set and $\{E_\alpha\}_{\alpha\in A}$ be a collection of sets that satisfy

- $\forall\alpha\in A\colon E_\alpha\subset K$;
- $\forall\alpha\in A\colon E_\alpha$ is closed;
- $\forall\alpha_1,\dots,\alpha_n\in A\colon \bigcap_{k=1}^n E_{\alpha_k}\neq\emptyset$.

Prove[2] that $\bigcap_{\alpha\in A} E_\alpha\neq\emptyset$.

---

[1] By definition, $f(A)=\{y\in Y\colon \exists x\in A, y=f(x)\}$ and $f^{-1}(B)=\{x\in X\colon f(x)\in B\}$
[2] Hint is available.