LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 4
2013–10–03 kl 10–12

1. Consider the operator $G\colon L^{2+} \to L^{2+}$ given by $y = Gu$

$$y(t) = \int_0^{+\infty} \Phi(t,s)u(s)\,ds.$$

a) Let P_t be the projection on the past up to the time t. Prove that G is causal if and only if $\Phi(t,s) = 0$ for $s > t$.

b) Let S_τ be the τ-shift forward operator on L^2, i.e. $(S_\tau x)(t) = x(t-\tau)$. Prove that G commutes with S_τ for all $\tau > 0$ if and only if $\Phi(t,s) = K(t-s)$ (i.e. Φ is a function that depends on the difference $t-s$ only).

2. Consider the system $x_{k+1} = 2x_k + u_k$, $y_k = x_k$ and the operator $y = Nu$.

a) Provide a causal matrix representation $y = Tu$ for N, taking $x_0 = 0$, and explain why it is not bounded operator from ℓ^{2+} to ℓ^{2+}.

b) Provide a matrix representation for N that gives a bounded linear operator from ℓ^{2+} to ℓ^{2+} and explain why it is not causal.

c) Provide a linear system that realises the operator in b) and discuss how it relates to $y_{k+1} = 2y_k + u_k$.

d) Take the input $u = \{1, 1/2, 1/4, 1/8, 1/16, \ldots\}$. Calculate the output $y \in \ell^{2+}$.

e) Calculate $\hat{u}$ in d), find the relation between $\hat{u}$ and $\hat{y}$ by performing $\mathcal{F}$-transform of the equation $y_{k+1} = 2y_k + u_k$ and calculate $y \in \ell^{2+}$ from the inverse $\mathcal{F}$-transform.

In 2e) two facts may be useful (easy to prove):

1. for any $|x| < 1$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots,$$
2. if f is analytical in the unit disc and $|a| < 1$ then
$$\frac{f(z)}{z-a} = \underbrace{\frac{f(z)-f(a)}{z-a}}_{P_+ \text{ part}} + \underbrace{\frac{f(a)}{z-a}}_{P_- \text{ part}}\ .$$