LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 3
2013–09–26 kl 10–12

1. Consider the Hamiltonian

$$H = \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}$$

where $R = R^T$ and $Q = Q^T$.

a) Prove that the eigenvalues of H are symmetric with respect to the imaginary axis.

b) Assume that H has no eigenvalues on the imaginary axis. Let $\begin{pmatrix} X \\ \Psi \end{pmatrix}$ be a basis of the stable invariant subspace (see Lecture 3, p. 8). Prove that if X is invertible then $P = \Psi X^{-1}$ is the stabilizing[1] solution to $A^T P + PA + PRP + Q = 0$.

2. Consider the system $\dot{x} = Ax + Bu$, $x(0) = x_0 \in \mathbf{R}^n$ where the matrix A is (Hurwitz) stable, the quadratic form

$$F(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q & S^T \\ S & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = x^T Q x + 2u^T S x + \|u\|^2$$

and the corresponding LQ control problem

$$J(x_0) = \min_{u \in \mathbf{L}^{2+}} \int_0^{+\infty} F(x(t), u(t))\, dt.$$

a) Complete the squares and calculate the u_{opt} and $J(x_0)$ under assumption that the operator Φ in the "u-square" term satisfies $\langle u, \Phi u\rangle \geq \epsilon \|u\|^2$, $\forall u \in \mathbf{L}^{2+}$.

b) Provide an interpretation for $\langle u, \Phi u\rangle \geq \epsilon \|u\|^2$ for all $u \in \mathbf{L}^{2+}$ in terms of the original system.

c) Provide a system interpretation for u_{opt} in style of Lecture 3, p. 6–7 and the corresponding Algebraic Riccati Equation (ARE) for P.

d) Show that the problem solved on the lecture $\min_{u \in \mathbf{L}^{2+}} \int_0^{+\infty} \|y(t)\|^2 + \|u(t)\|^2\, dt$ has the same ARE and u_{opt} even *without* the assumption that A is stable. Hint: perform the feedback change (Remark on page 4) and use the result in c).

3. Consider the operator $N\colon u(\cdot) \to y(\cdot)$ where

$$\begin{cases} \dot{x} = Ax + Bu, & x_0 = 0, \\ y = Cx \end{cases}$$

with (A, B) being stabilizable[2] and (C, A) detectable[3]. Prove that $N \in L(L^{2+}, L^{2+})$ if and only if A is stable.

4. All other problems from Exercise 1 and 2 that you did not have time yet to look at.

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[1] It means $A + RP$ is (Hurwitz) stable.
[2] There exists F such that $A + BF$ is stable.
[3] That is (A^T, C^T) is stabilizable.