LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 2
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1. Consider the system

$$\begin{cases} \dot{x} &= Ax, \quad x(0) = x_0 \in \mathbf{R}^n, \\ y &= Cx, \end{cases}$$

where the matrix A is (Hurwitz) stable, and the operator $y(\cdot) = Mx_0$. Show that $M^*M = P$ is the solution to the Lyapunov equation $A^TP + PA + C^TC = 0$ (observability Gramian). Solve the problem by

a) calculating the system that represents M^*M in style of Lecture 2,

b) derivating $x(t)^TC^TCx(t)$, expressing P as an integral and showing that $P = M^*M$.

2. Consider the system

$$\begin{cases} x_{k+1} &= Ax_k + Bw_k, \quad x_0 = 0, \\ y_k &= Cx_k + Dw_k, \end{cases}$$

where the $n \times n$ matrix A is (Schur) stable (i.e. for all eigenvalues $|\lambda_i| < 1$).

a) Prove that the operator $N\colon \ \{w_k\}_{k=0}^{\infty} \to \{y_k\}_{k=0}^{\infty}$ belongs to $L(\ell^{2+}, \ell^{2+})$.

b) Calculate N^* and provide a linear system that represents it.

3. Let X, Y be normed vector spaces and $A \in L(X, Y)$.

a) Prove that $\|A^*\| \le \|A\|$ (and thus $A^* \in L(Y^*, X^*)$).

b) Prove that if X and Y are reflexive (i.e. $X = X^{**}$ and $Y = Y^{**}$) then $A^{**} = A$.

4. Consider the system $\dot{\psi} = A\psi + w$ on $[0, +\infty)$ with anti-stable matrix A. Show that for all $w \in L^{2+}$ there exists a unique solution $\psi \in L^{2+}$ and this is the only solution that satisfies $\psi(+\infty) = 0$. Find an explicit integral formula for such ψ.

5. Consider $f \in L^1(\mathbf{R})$ and the convolution operator $Tg = f * g$. i.e.

$$Tg(t) = \int_{\mathbf{R}} f(t-s)g(s)\,ds.$$

a) Prove that $T \in L(L^p, L^p)$, for all $1 \le p \le +\infty$ using Riesz-Thorin interpolation theorem.

b) Prove the lemma (Lecture 2, p. 3) using a).

6. Let H be a Hilbert space and $A \in L(H, H)$. Prove that

a) $\mathrm{Ker}(A^*) = \mathrm{Im}(A)^\perp$.

b) $\mathrm{Ker}(A)^\perp = \mathrm{closure}(\mathrm{Im}(A^*))$.

Here $\perp$ means the orthogonal complement. Definitions for Ker and Im are on page 6, Lecture 2.