LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 13
2013–12–02 kl 13–15

1. Let X be a real normed vector space, $K \subset X$ be convex and $h(f) = \sup_{k \in K} \langle f, k \rangle$.

a) Show that the set $K^* = \{f \in X^* \colon h(f) < +\infty\}$ is a convex cone.

b) Show that if there exists an optimal point $k_0 \in K$ for the primal problem

$$\inf_{k \in K} \|x - k\| = \sup_{f \in BX^*} (\langle f, x \rangle - h(f))$$

then the dual optimal f_0 is aligned with $x - k_0$.

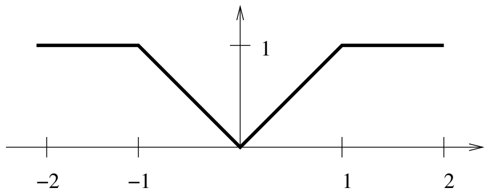
c) Explain how 1b) can be used to define a notion of "orthogonal projection onto a convex set" in a normed vector space.

2. Consider a real normed vector space X. Let $f_1, \ldots, f_n \in X^*$ and $c_1, \ldots, c_n \in \mathbf{R}$ be given and define $K = \{x \in X \colon \langle f_k, x \rangle \le c_k,\ k = 1, \ldots, n\}$. Show that $K \neq \emptyset$ if and only if

$$\forall \lambda_k \ge 0 \colon \sum_{k=1}^{n} \lambda_k f_k = 0 \quad \Rightarrow \quad \sum_{k=1}^{n} \lambda_k c_k \ge 0.$$

3. Let X be a real normed vector space and $K \subset X$ be a closed set with $\mathrm{int}(K) \neq \emptyset$. Prove that K is convex if and only if for any $x_0 \in \partial K$[1] there exists a non-zero $f \in X^*$ such that $\langle f, x_0 \rangle \ge \langle f, x \rangle$ for all $x \in K$.

4. Let F be a function given below. Calculate F^* and F^{**}. Explain how F and F^{**} are related.



[1] Boundary of K.