LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 12
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1. **a)** Let X be a normed vector space. Show by Hahn-Banach theorem that for any $x \in X$ there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

b) Prove that $\|x\| = \sup_{f \in BX^*} |f(x)|$.

c) Let X and Y be Banach spaces and $A \colon X \to Y$ be a bounded linear operator. Prove that $\|A\| = \|A^*\|$.

2. Let X be a normed vector space and $H \subset X$.

a) Prove that H is a hyperplane if and only if there exists a nonzero linear functional f such that $H = \{x \in X \colon f(x) = \text{const}\}$.

b) Prove that if H is a hyperplane that does not contain the origin then there is a unique linear functional f such that $H = \{x \in X \colon f(x) = 1\}$.

c) Prove that H is a closed hyperplane if and only if there exists a nonzero $f \in X^*$ such that $H = \{x \in X \colon f(x) = \text{const}\}$.