1. Let $c_1, \ldots, c_n \in \mathbf{C}$ be complex numbers. Consider the problem

$$M = \sup_{f \in B\mathbf{H}^\infty} |\sum_{k=0}^{n} c_k f_k|$$

where $f(z) = \sum_{k=0}^{\infty} f_k z^k$.

a) Show that the dual problem is

$$M = \inf_{g \in \mathbf{H}_0^1} \|p - g\|_1$$

where $p(z) = \sum_{k=0}^{n} c_k z^{-k}$ and it is equivalent to the minimum norm problem

$$M = \inf\{\|h\|_1 \colon h \in \mathbf{H}^1,\, h(z) = c_n + c_{n-1}z + \ldots + c_0 z^n + \ldots\}.$$

b) Show that both problems have unique solutions[1] f_0 and g_0 respectively. Moreover, $f_0(p - g_0) = |p - g_0|$ on the unit circle.

c) Show that

$$f_0(p - g_0) = Cz^{-m} \prod_{k=1}^{m} (z - \alpha_k)(1 - \bar{\alpha}_k z)$$

where $0 \le m \le n$, $C > 0$, $0 < |\alpha_k| \le 1$. After re-indexing α_k, there exists $r \le m$ such that $0 < |\alpha_k| < 1$ for $k \le r$ and

$$\begin{aligned} f_0(z) &= \gamma z^{n-m} \prod_{k=1}^{r} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \\ p(z) - g_0(z) &= C\bar{\gamma} z^{-n} \prod_{k=1}^{m} (1 - \bar{\alpha}_k z)^2 \prod_{k=r+1}^{m} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \end{aligned}$$

where $|\gamma| = 1$.

d) For small $|z|$ write $(c_n + c_{n-1}z + \ldots + c_0 z^n)^{1/2} = \sum_{k=0}^{\infty} \lambda_k z^k$ and set $R_n(z) = \sum_{k=0}^{n} \lambda_k z^k$. Assume that $R_n(z) \neq 0$ for $|z| < 1$. Then $\exists \alpha_k$, $0 < |\alpha_k| \le 1$ such that $R_n(z) = \lambda_0 \prod_{k=1}^{m} (1 - \bar{\alpha}_k z)$. Then $z^{-n} R_n(z)^2 = c_n z^{-n} \prod_{k=1}^{m} (1 - \bar{\alpha}_k z)^2$ has the form $p - g$, $g \in \mathbf{H}^1$. Setting

$$f_0(z) = \frac{\bar{c_n}}{|c_n|} z^{n-m} \prod_{k=1}^{m} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}$$

gives the solution and $M = \sum_{k=0}^{n} |\lambda_k|^2$. Explain everything!

[1] Note that you need to prove *two* things: existence and uniqueness