LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 10
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1. Let X be a Hilbert space and M be a closed linear subspace. Write down the duality for $\inf_{m \in M} \|x - m\|$ and provide a geometrical interpretation of it for $X = \mathbf{R}^3$ and $M = \mathbf{R}$.

2. Let X be a real normed vector space and $g_i \in X^*$ for $i = 1, \ldots, n$.

a) Prove that g_i are linearly independent if and only if $\operatorname{Im}(g_1, \ldots, g_n) = \mathbf{R}^n$.

b) Let $f \in X^*$ be given and assume g_i are linearly independent. Consider the optimization problem

$$\inf_{\lambda_i \in \mathbf{R}} \|f - \sum_{i=1}^{n} \lambda_i g_i\|.$$

Find the dual problem.

c) Prove that if $f(x) = 0$ for all $x \in X$ such that $g_i(x) = 0$, $i = 1, \ldots, n$ then there exist $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ such that $f = \sum_{i=1}^{n} \lambda_i g_i$.

d) Let g_i be again linearly independent and $\alpha_i \in \mathbf{R}$ be given, $i = 1, \ldots, n$. Denote $S = \{x \in X | g_i(x) = \alpha_i\}$ and consider the problem

$$\inf_{x \in S} \|x\|.$$

Explain how the problem can be solved via a finite-dimensional optimization by using the duality.