LUND UNIVERSITY
DEPARTMENT OF MATHEMATICS

FAN IN CONTROL
Exercise 1
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1. a) Prove that all solutions to the linear differential equation $\dot{x}(t) = A(t)x(t)$ build a vector space.

b) Consider the map $w \to y$

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)w(t), \\ y(t) &= C(t)x(t) + D(t)w(t). \end{cases}$$

What choice of the initial state $x(0) = x_0$ makes it be a linear map?

c) Consider a linear differential equation $\dot{x}(t) = Ax(t)$, $x(0) = x_0$, $t \in [0, +\infty)$ with a stable matrix A. Prove that the map $x_0 \to x(\cdot)$ is a continuous linear operator from $\mathbf{R}^n$ to L^{2+}.

2. Denote

$$c_0 = \{x = (x_0, x_1, x_2, \ldots) \mid \lim_{k\to+\infty} x_k = 0\},$$

i.e. the set of all sequences that converges to 0.

a) Prove that c_0 is a vector space and $\|x\| = \max_k |x_k|$ is a norm on c_0.

b) Prove that $c_0^* = \ell^1$.

c) Prove that $(\ell^1)^* = \ell^\infty$.

3. Let X, Y and Z be normed vector spaces and $A \in L(X, Y)$. The induced (operator) norm is defined as $\|A\| = \sup_{\|x\|\leq 1} \|Ax\|$. Prove that

a) $\|A\|$ is a norm.

b) $\|Ax\| \leq \|A\|\|x\|$, $\forall x \in X$.

c) $\|BA\| \leq \|B\|\|A\|$, $\forall A \in L(X, Y), B \in L(Y, Z)$.

d) $\|A\| = \sup_{\|x\|=1} \|Ax\|$, i.e. the supremum can be taken over the unit *sphere* only.

4. *Cauchy-Schwarz inequality* for a Hilbert space H is $|\langle x, y\rangle| \leq \|x\|\|y\|$, $\forall x, y \in H$.

a) Prove Cauchy-Schwarz inequality for a real Hilbert space.

b) Prove Cauchy-Schwarz inequality for a complex Hilbert space.

c) Prove that the function $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on H.

d) Prove that $(\ell^2)^* = \ell^2$.

5. Prove Riesz representation theorem for a Hilbert space H:
Let H be a Hilbert space. Then for any $f \in H^*$ there exists a unique $g \in H$ such that $f(x) = \langle g, x\rangle$, $\forall x \in H$. Moreover $\|f\| = \|g\|$.

6. Prove that *if X is a normed vector space then X^* is Banach space.*

(Proofs in Problems 5 and 6 can be found in the literature — read and understand them.)