A Critical Configuration for Reconstruction from Rectilinear Motion

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Abstract

This paper investigates critical configurations for projective reconstruction from multiple images taken by a camera moving in a straight line. Projective reconstruction refers to a determination of the 3D geometrical configuration of a set of 3D points and cameras, given only correspondences between points in the images. A configuration of points and cameras is critical if it can not be determined uniquely (up to a projective transform) from the image coordinates of the points.

It is shown that a configuration consisting of any number of cameras lying on a straight line, and any number of points lying on a twisted cubic constitutes a critical configuration. An alternative configuration consisting of a set of points and cameras all lying on a rational quartic curve exists.

1 Introduction

When is the solution to the 3D reconstruction problem from images unique? Despite efforts for more than half a century, nobody has been able to give a complete answer [5, 4]. This paper is a step forward in revealing the full picture for any number of images.

A configuration of 3D points P_i and cameras P^j is said to be critical if they can not be reconstructed uniquely from the projected image points $x_i^j = P^j X_i$. In this context, two reconstructions are considered to be equivalent if they differ by a projective transformation of space. In other words a configuration is critical if there exists an inequivalent alternative *conjugate* configuration consisting of cameras Q_i and Q^j such that $P^j P_i = Q^j Q_i$ for all i, j .¹

Critical configurations for the two-view case are well understood – a configuration is critical only if all the points and cameras lie on a ruled quadric.² However, for three or more views the situation is more complex. The most important re-

sult in this direction is contained in [3], in which it is shown that a configuration of any number of points and cameras lying on the intersection of two ruled quadrics is critical. Other work in this area includes [2] which forms a basis for the work of [3], and [6] which identifies critical configurations involving just six points.

Although [3] identifies a large class of critical configurations, it is shown in this paper that a further family of critical configurations exists for multiple views. One of the reasons for obtaining a complete enumeration is to show that in fact they are rather rare and that in the end, perhaps we do not need to worry about them. This is in contrast to two-view configurations where the critical ones may cause problems.

2 Example of an ambiguous configuration

Consider a pair of rational curves in projective 3-space, $\mathcal{P}^3$: $P(\theta) = (1, \theta, \theta^2, \theta^3)^\top$ and $Q(\theta) = (\theta, \theta^2 - \theta^3, \theta^4, 1)^\top$. The first of these curves is a cubic curve (often called a *twisted cubic*), and the second is a *quartic curve*, since it contains a fourth power of the parameter θ .

Let λ be a variable parameter, and let

$$P^\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda - \lambda^2 & 1 - \lambda & -1 & 0 \\ \lambda^3 & \lambda^2 & \lambda & 1 \end{bmatrix}$$

and

$$Q^\lambda = \begin{bmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1 & 0 & -\lambda^2 + \lambda^3 \\ 0 & 0 & 1 & -\lambda^4 \end{bmatrix}$$

be a pair of camera matrices. The camera centres of these two cameras trace out trajectories parametrized by λ , as follows

$$C_P(\lambda) = (0, -1, \lambda - 1, \lambda)^\top \text{ and } C_Q(\lambda) = (\lambda, \lambda^2 - \lambda^3, \lambda^4, 1)^\top.$$

We see that the first camera centre moves along a straight line, whereas the second camera centre moves along the quartic curve $Q(\theta)$.

¹We ignore the trivial ambiguity that arises from points lying on a line containing all the camera centres – the location of such points may vary along this base line.

²The converse is also true, except for certain special cases discovered recently ([4]).

Projecting the points on the two space curves into the two images leads to

$$\begin{aligned}(\theta - \lambda)P^\lambda P(\theta) &= Q^\lambda Q(\theta) \\ &= (\theta - \lambda, \theta^2 - \lambda^2 - (\theta^3 - \lambda^3), \theta^4 - \lambda^4)^\top.\end{aligned}$$

Ignoring the insignificant scale factor $\theta - \lambda$, this says that $P^\lambda P(\theta) = Q^\lambda Q(\theta)$, which means that the images of the two curves in the respective cameras are the same. Thus in this example, a camera moving in a straight line viewing a twisted cubic gives the same image as a camera moving along a quartic curve viewing its own trajectory curve. This is an example of reconstruction ambiguity. From the set of image measurements and correspondences between the successive images it is impossible to compute the camera motion and scene geometry unambiguously.

2.1 Remarks on this example

This example of ambiguity is quite distinct from the critical configurations discussed in [3], and has many curious properties. At first sight, it may appear that it is just a special case of the [3] examples, in which the twisted cubic and line in the first $(P(\theta))$ configuration are simply the intersection of two ruled quadrics, and the quartic curve in the conjugate configuration, containing points $Q(\theta)$ is likewise the intersection of two distinct ruled quadrics – an elliptic quartic curve. However, that is not the case as will be seen later. In particular it will be shown that the curve $Q(\theta)$ lies on a *unique* quadric, and similarly, the twisted cubic containing points $P(\theta)$ and the line of centres of the cameras P^λ are not in the intersection of two quadrics.

The critical quadrics. Given a critical configuration with more than two cameras, such as the one given in this example, one may derive 2-view critical configurations consisting of the full point set and two chosen cameras. According to the known classification of critical configurations for two views, the set of points and the two camera centres must all lie on a ruled quadric surface, S . To be more precise, let (P^i, P^j) be a pair of cameras and (Q^i, Q^j) the corresponding pair of cameras in the conjugate configuration. It was shown in [2] that the critical quadric is represented by a matrix $S_p^{ij} + S_p^{ij\top}$, where S_p^{ij} is a 4×4 matrix

$$S_p^{ij} = P^i{}^\top F_q^{ij} P^j \quad (1)$$

and F_q^{ij} is the fundamental matrix of the camera pair (Q^i, Q^j) . In this paper, we will represent the quadric by the asymmetric matrix S_p^{ij} .

For a multiple-view critical configuration, we obtain a (possibly) different quadric S_p^{ij} for each choice of the two

cameras P^i and P^j . The set of points in the critical configuration must lie on the intersection of all the quadrics. For example, for a three-view critical configuration, the set of critical points must lie in the intersection of the three quadrics $S_p^{10} \cap S_p^{20} \cap S_p^{21}$. Similarly in the conjugate configuration, all points must lie in the intersection of the three quadrics S_q^{ij} .

Let us compute the critical quadrics in the case of the example of this section. We choose two values λ_0 and λ_1 of the parameter λ , and compute the critical quadrics according to (1). The computation is straight-forward, leading to the formula given in Fig 1. Note the fact that S_q is independent of the choice of λ_0 and λ_1 . In other words for all choices of pairs of cameras $(P^{\lambda_0}, P^{\lambda_1})$ and corresponding conjugate cameras $(Q^{\lambda_0}, Q^{\lambda_1})$, the critical quadric $S_q^{\lambda_1 \lambda_0}$ is the same.

It is easily verified that the complete curve $Q(\theta)$ lies on the quadric S_q , and the curve $P(\theta)$ lies on each $S_p^{\lambda_1 \lambda_0}$, in agreement with the general theory. In addition, the camera centres $C_p(\lambda_0)$ and $C_p(\lambda_1)$ lie on the quadric $S_p^{\lambda_1 \lambda_0}$, as they must. However these are the only two points of intersection of the line of camera centres $C_p(\lambda)$ and the quadric $S_p^{\lambda_1 \lambda_0}$. Since a quadric is defined uniquely by a twisted cubic and two points (see Proposition 4.7 below) we may state

- $S_p^{\lambda_1 \lambda_0}$ is the unique quadric that contains the twisted cubic $P(\theta)$ and the two camera centres $C_p(\lambda_1)$ and $C_p(\lambda_0)$. Each different pair of camera centres $(C_p(\lambda_1), C_p(\lambda_0))$ on the straight line $C_p(\lambda)$ defines a different quadric.

Why are all $S_q^{\lambda_1 \lambda_0}$ the same? We examine this example from the point of view of the Q -configuration. It was noted that all the critical quadrics S_q are the same. This situation was investigated in [2] for the three-view case, where it was concluded that if all three quadrics are the same, then they do not form a critical set. The present example gives a more complete understanding of what happens in this situation.

It was shown in [2] that for a general three view configuration (cameras numbered 0, 1, 2), the intersection of the three quadrics $S_q^{10} \cap S_q^{20} \cap S_q^{21}$ forms a critical set, except for those points lying on the intersection of three special planes. See [2] Theorem 1 and Corollary 1 for details (with the role of the P and Q configurations reversed). The exception to this is the case where the camera centres of P^0 , P^1 and P^2 are collinear. It was also shown in [2] that this will always occur when S_q^{10} , S_q^{20} and S_q^{21} are all the same, which is the case in the present example. Consequently, we are unable to conclude that S_q forms a critical set.

In conformance with this finding the whole of the ruled quadric S_q indeed does *not* form a critical set in the current example. However, as this example shows, the curve $C_q(\lambda)$ which lies on S_q *does* form a critical set in this instance.

$$S_p^{\lambda_1 \lambda_0} = \begin{bmatrix} 0 & 0 & \lambda_0 + \lambda_1 - \lambda_0 \lambda_1 & 1 - \lambda_0 - \lambda_1 \\ 0 & -2(\lambda_0 + \lambda_1 - \lambda_0 \lambda_1) & -1 + \lambda_0 + \lambda_1 & -1 \\ \lambda_0 + \lambda_1 - \lambda_0 \lambda_1 & -1 + \lambda_0 + \lambda_1 & 2 & 0 \\ 1 - \lambda_0 - \lambda_1 & -1 & 0 & 0 \end{bmatrix} \quad (2)$$

and

$$S_q = S_q^{\lambda_1 \lambda_0} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}. \quad (3)$$

Figure 1: Critical quadrics for the example in section 2.

Goal of the paper. In the rest of this paper we will generalize this example by showing that it is typical of a family of ambiguous reconstructions. In particular, the following results will be shown.

Theorem 2.1.

1. Any configuration of points lying on a twisted cubic and cameras lying on a straight line is critical.
2. Any configuration of points and cameras all lying on a parametrized quartic curve is critical.

These two types of configuration are conjugate, in that a configuration of one type always has a conjugate configuration of the other type.

It is possible that each of the configurations mentioned in this result may have other critical configurations. However it seems very unlikely that this is the case, provided there are sufficiently many points (at least seven) and cameras (at least three) involved. A partial proof of this is provided.

This analysis is part of an ongoing investigation, to which several other authors have contributed ([6, 2, 3]) to determine all possible critical configurations involving at least three views. It will be shown how this result fits into this classification effort.

3 Rational curves

To understand the example of section 2 thoroughly, we need to study the properties of parametrized curves. This section will be concerned with parametrized curves in projective spaces of dimension 2 (the image) and dimension 3 (the world), and their relationship to each other. Traditionally, parametrized curves have been called *rational* curves ([7]), and in future, that is the word that will be used in this paper. We begin by defining such curves, and stating or deriving some of their elementary properties.

Definition 3.2. A rational curve of degree d in $\mathcal{P}^n$ is the locus of points of the form $\mathbf{X}(\theta) =$

$(a_0(\theta), a_1(\theta), \dots, a_n(\theta))^\top$ where each $a_i(\theta)^\top$ is a real polynomial of degree at most d , at least one $a_i(\theta)$ has degree d , and the $a_i(\theta)$ do not have a common factor of degree greater than zero.

For brevity we will often use the word *curve* to mean a *rational curve*. A rational curve in $\mathcal{P}^n$ may be written in the form $\mathbf{A}(1, \theta, \theta^2, \dots, \theta^d)^\top$ where $\mathbf{A}$ is the $(n+1) \times (d+1)$ matrix of coefficients of the several polynomials $a_i(\theta)$. We denote the vector $(1, \theta, \theta^2, \dots, \theta^d)^\top$ by $\Theta^{(d)}$. With this notation, the rational curve is the locus of points

$$\mathbf{X}(\theta) = \mathbf{A}\Theta^{(d)}.$$

Normally, we will require that the matrix $\mathbf{A}$ representing the curve has full row-rank, that is, rank $n+1$. This is the requirement that the curve not lie in a subspace (hyperplane) in $\mathcal{P}^n$. For if π represents a hyperplane containing points $\mathbf{A}\Theta^{(d)}$ for all θ , then $\pi^\top \mathbf{A}\Theta^{(d)} = 0$ for all θ , so π lies in the left null space of $\mathbf{A}$, and conversely. A curve that does not satisfy this rank condition is said to be *degenerate*.

Projective transformations of curves. For the most part, we will be concerned with properties of curves that are preserved under projective transforms. Let $\mathbf{H}$ be a projective transformation of $\mathcal{P}^n$, and let $\mathbf{A}\Theta^{(d)}$ represent a rational curve. Applying $\mathbf{H}$ to each point of the curve transforms it to a new curve represented by the matrix $\mathbf{A}' = \mathbf{H}\mathbf{A}$. It is easily seen that the new curve $\mathbf{A}'\Theta^{(d)}$ is also a rational curve of degree d . Note that this requires checking that the resulting polynomials $a'_i(\theta)$ have no common factor, and that one of them still has degree d .

Note that multiplication of $\mathbf{A}$ on the left by $\mathbf{H}$ corresponds to applying a sequence of elementary row operations³ to $\mathbf{A}$, and that conversely any sequence of elementary row operations constitutes a projective transformation of the curve. One may apply a sequence of elementary row operations to $\mathbf{A}$ to put it in row-echelon form, resulting in the following form for a rational curve.

³Elementary row operations consist of permuting rows, adding a multiple of one row to another, or multiplying a row by a non-zero constant.

Proposition 3.3. *A rational curve of degree d is projectively equivalent to a curve of the form $(a_0(\theta), a_1(\theta), \dots, a_n(\theta))^\top$, where $\text{degree}(a_0) < \text{degree}(a_1) < \dots < \text{degree}(a_n) = d$.*

4 The twisted cubic

Of particular interest is the case of a rational cubic curve in $\mathcal{P}^3$, known as the twisted cubic. Properties of the twisted cubic are given here for convenience only. More information on the twisted cubic may be found in [7];

Definition 4.4. A twisted cubic is the locus of points of the form $A\Theta^{(3)}$ where A is a 4×4 invertible matrix.

Since A is invertible, it follows that all twisted cubics are projectively equivalent. A twisted cubic has 12 degrees of freedom (after accounting for reparametrization), and so we expect it to be defined by 6 points (each point that lies on the twisted cubic contributing two constraints). This argument will now be made more explicit. First of all, observe that no four points on a twisted cubic can be coplanar. This is true, since any twisted cubic is equivalent to the one given by $\mathbf{X}(\theta) = (1, \theta, \theta^2, \theta^3)^\top$. Let $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)^\top$ be a plane. The intersection of the plane and the cubic consists of all points satisfying $0 = \pi^\top \Theta^{(3)} = \pi_0 + \pi_1\theta + \pi_2\theta^2 + \pi_3\theta^3$, which is satisfied for at most three values of θ . We may now state:

Proposition 4.5. *Let $\mathbf{X}_1, \dots, \mathbf{X}_6$ be six points in space, no four of which are coplanar. Then there exists a unique twisted cubic passing through all six $\mathbf{X}_i$.*

The construction given in Semple and Kneebone ([7]) involves the Carlsson map ([1]), though (of course) not under that name. A specific construction of the required twisted cubic is given as follows:

- *The twisted cubic through six points $\mathbf{X}_1, \dots, \mathbf{X}_6$ is given by $\Gamma(\Gamma(\mathbf{X}_5) + \theta\Gamma(\mathbf{X}_6))$, where Γ is a Carlsson map with respect to the reference tetrahedron with vertices $\mathbf{X}_1, \dots, \mathbf{X}_4$.*

4.1 Twisted cubics and quadric surfaces

Since all twisted cubics are projectively equivalent, and the cubic $(1, \theta, \theta^2, \theta^3)^\top$ lies on the ruled quadric surface $XT = YZ$, it follows that every twisted cubic lies on a ruled quadric surface. A quadric surface has 9 degrees of freedom, and there exists a unique quadric passing through 9 general points. Since a unique twisted cubic may be chosen to pass through 6 points, one might guess that there exists a unique quadric containing a given twisted cubic and three general points. However, this is false. The truth is that a quadric is defined by a twisted cubic, plus **two** points, as we will show

presently. Thus, paradoxically, a twisted cubic acts as seven points, not six, as far as constraining the quadric surface is concerned.

We begin by determining the family of quadrics containing a given twisted cubic.

Theorem 4.6. *The family of quadrics containing a given twisted cubic is a 2-parameter family (often called a net). Any such quadric may be written as a linear combination of three quadrics: $S = \alpha S_0 + \beta S_1 + \gamma S_2$. Let $\mathbf{X}_1, \dots, \mathbf{X}_7$ be 7 arbitrary distinct points on the twisted cubic, then any quadric S contains all seven $\mathbf{X}_i$ if and only if it contains the complete twisted cubic.*

The reason we call this a 2-parameter family, rather than a 3-parameter family is of course because the overall scale is immaterial.

Proof. Since all twisted cubics are equivalent, we consider the curve $\mathbf{X}(\theta) = \Theta^{(3)}$. Let S be a symmetric matrix representing a quadric. The condition that $\mathbf{X}(\theta_i)$ lie on the quadric S gives an equation $\Theta_i^{(3)\top} S \Theta_i^{(3)} = 0$. This equation can be expanded out in terms of the diagonal and above-diagonal entries of S to give

$$S_{11} + \theta_i(2S_{12}) + \theta_i^2(2S_{13} + S_{22}) + \theta_i^3(2S_{14} + 2S_{23}) + \theta_i^4(2S_{24} + S_{33}) + \theta_i^5(2S_{34}) + \theta_i^6(S_{44}) = 0.$$

Since this is a polynomial equation in θ of degree 6, it can not have more than 6 solutions. Consequently, given seven values θ_i that satisfy this equation, it must follow that the polynomial is identically zero. Hence

$$\begin{aligned} S_{11} &= 2S_{12} = 2S_{13} + S_{22} = 2S_{14} + 2S_{23} = 2S_{24} + S_{33} \\ &= 2S_{34} = S_{44} = 0. \end{aligned}$$

The family of quadrics containing the twisted cubic $(1, \theta, \theta^2, \theta^3)^\top$ is generated by the three matrices

$$S_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, we consider the determination of a quadric by a twisted cubic and two points.

Proposition 4.7. *There exists a unique quadric surface containing a given twisted cubic and two other general points.*

Proof. The set of quadrics containing the standard twisted cubic $(1, \theta, \theta^2, \theta^3)^\top$ is generated by the three quadrics S_0 , S_1 and S_2 above. If an extra point $\mathbf{X}$ is added, not lying on the twisted cubic, then it introduces a further linear constraint, thereby reducing the family of solutions to a one-parameter family, generated by two quadrics. Addition of a second point not on the intersection of the two quadrics further reduces the set of possible solutions to a single quadric.

4.2 Twisted cubic plus a line

Next, we consider a twisted cubic and a straight line. It is well known that a twisted cubic plus a line may occur as the intersection of two quadrics. As an example, the line $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})^\top = (\theta, 1, \theta, 1)^\top$ and the twisted cubic $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})^\top = (\theta, \theta^2, \theta^3, 1)^\top$ are the intersection of the two quadrics $\mathbf{z}\mathbf{t} = \mathbf{x}\mathbf{y}$ and $\mathbf{y}^2 + \mathbf{x}^2 = \mathbf{x}\mathbf{z} + \mathbf{y}\mathbf{t}$. However, not all twisted cubic + line pairs can occur as such an intersection. The required condition is given next (though the proof is omitted, see [7]).

Theorem 4.8. *A twisted cubic and straight line are the intersection of two quadrics if and only if they meet in two points (possibly complex). A point of common tangency counts as two intersections.*

As an example of complex intersection, the line $(\theta, -1, -\theta, 1)^\top$ and the twisted cubic $(\theta, \theta^2, \theta^3, 1)^\top$ are the intersection of the two quadrics $\mathbf{z}\mathbf{t} = \mathbf{x}\mathbf{y}$ and $\mathbf{y}^2 - \mathbf{x}^2 = \mathbf{x}\mathbf{z} - \mathbf{y}\mathbf{t}$. They meet in the point $(i, -1, -i, 1)^\top$ and its complex conjugate.

Thus, most twisted cubic and line pairs do not lie on the intersection of two quadrics.

5 Quartic curves

A non-degenerate quartic (degree 4) curve in $\mathcal{P}^3$ is represented by a 4×5 matrix $\mathbf{A}$, of rank 4.

5.1 Rational and elliptic quartics.

This paper considers critical configurations made up of rational quartics, whereas [3] considered configurations occurring as the intersection of two quadrics, called *elliptical quartics*. We now indicate that these are two quite different types of curve.

Some elliptic quartics are rational. As an example, the two quadrics $\mathbf{z}\mathbf{t} = \mathbf{x}\mathbf{y}$ and $\mathbf{x}^2 + \mathbf{y}^2 - \mathbf{t}^2 = 0$ meet in the curve

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})^\top = (\theta^4 - 1, 2\theta(\theta^2 + 1), 2\theta(\theta^2 - 1), (\theta^2 + 1)^2)^\top.$$

Not all elliptic quartics are rational. In order to be parametrizable, the real part of the curve must be connected.

However, it is easy to find two ruled quadrics that do not have connected intersection.

Not all rational quartics are elliptic. Consider a rational quartic given by $\mathbf{A}\Theta^{(4)}$, where $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} 1 & & & p \\ & 1 & & q \\ & & 1 & r \\ & & & 1 & s \end{bmatrix}.$$

One may choose 9 values of θ (for instance the integers between -4 and 4) and verify by direct computation that for generic values of p, q, r and s , these 9 points lie on a unique quadric. In fact, direct computation shown that this will be true, unless $-q^2 + pr + r^3 - 2qrs + ps^2 = 0$. In this latter case there will be a pencil of quadrics containing the quartic curve. Thus, most rational quartic curves lie on a unique quadric.

6 General classes of critical configurations

We now prove the main result of this paper, namely that the two types of configuration given in Theorem 2.1 are always critical. The result follows directly from the following Theorem 6.9.

Theorem 6.9. *Let $\mathbf{A}\Theta^{(3)}$ and $\mathbf{A}'\Theta^{(4)}$ be non-degenerate cubic and quartic curves in $\mathcal{P}^3$. Let $\mathbf{Q}$ be a camera with centre lying on the quartic curve $\mathbf{A}'\Theta^{(4)}$. Then there exists a camera $\mathbf{P}$ such that $\mathbf{P}\mathbf{A}\Theta^{(3)} = \mathbf{Q}\mathbf{A}'\Theta^{(4)}$, for all θ . Furthermore, as the camera $\mathbf{Q}$ moves along the quartic curve, the locus of the camera centre of $\mathbf{P}$ is a straight line, determined only by the two curves.*

If the cubic curve $\mathbf{A}\Theta^{(3)}$ is fixed, then for varying choices of the quartic curve $\mathbf{A}'\Theta^{(4)}$, this straight line ranges over all possible lines in space.

Proof. We are free to choose a coordinate frame in such a way that the cubic curve is equal to $\Theta^{(3)} = (1, \theta, \theta^2, \theta^3)^\top$, that is $\mathbf{A}$ is the identity matrix. Denote the vector $(1, \lambda, \dots, \lambda^4)$ by $\Lambda^{(4)}$. Now let $\mathbf{Q}$ be a camera with centre lying on the curve $\mathbf{A}'\Theta^{(4)}$. Thus, for some value of a parameter λ , the point $\mathbf{A}'\Lambda^{(4)}$ is the camera centre for $\mathbf{Q}$, which means $\mathbf{Q}\mathbf{A}'\Lambda^{(4)} = \mathbf{0}$. Now define $\mathbf{P} = \mathbf{Q}\mathbf{A}'\mathbf{G}$, where $\mathbf{G}$ is the 5×4 matrix

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & & & \\ \lambda & 1 & & \\ \lambda^2 & \lambda & 1 & \\ \lambda^3 & \lambda^2 & \lambda & 1 \end{bmatrix}.$$

It will be shown that $\mathbf{P}\Theta^{(3)} = \mathbf{Q}\mathbf{A}'\Theta^{(4)}$ for all θ . First, we compute

$$\mathbf{G}\Theta^{(3)} = \begin{pmatrix} 0 \\ 1 \\ \lambda + \theta \\ \lambda^2 + \lambda\theta + \theta^2 \\ \lambda^3 + \lambda^2\theta + \lambda\theta^2 + \theta^3 \end{pmatrix}$$

from which it follows that

$$(\theta - \lambda)\mathbf{G}\Theta^{(3)} = \begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix} - \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{pmatrix}.$$

Consequently,

$$(\theta - \lambda)\mathbf{P}\Theta^{(3)} = (\theta - \lambda)\mathbf{Q}\mathbf{A}'\mathbf{G}\Theta^{(3)} = \mathbf{Q}\mathbf{A}'\Theta^{(4)} - \mathbf{Q}\mathbf{A}'\Lambda^{(4)} = \mathbf{Q}\mathbf{A}'\Theta^{(4)}.$$

The last step follows from the fact that $\mathbf{Q}\mathbf{A}'\Lambda^{(4)} = \mathbf{0}$. Hence, ignoring the insignificant scale factor $\theta - \lambda$ we see that $\mathbf{P}\Theta^{(3)} = \mathbf{Q}\mathbf{A}'\Theta^{(4)}$ as required.

Next, we compute the camera centre of $\mathbf{P}$. Let $\mathbf{n} = (n_0, n_1, \dots, n_4)^\top$ be the generator of the (right) null-space of $\mathbf{A}'$. Then we show that the camera centre of $\mathbf{P}$ is the point

$$\mathbf{C}_P = \lambda(n_0, n_1, n_2, n_3)^\top - (n_1, n_2, n_3, n_4)^\top,$$

which traces out a straight line as λ varies. Compute first

$$\begin{aligned} \mathbf{G}\mathbf{C}_P &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & & & \\ \lambda & 1 & & \\ \lambda^2 & \lambda & 1 & \\ \lambda^3 & \lambda^2 & \lambda & 1 \end{bmatrix} \begin{pmatrix} \lambda n_0 - n_1 \\ \lambda n_1 - n_2 \\ \lambda n_2 - n_3 \\ \lambda n_3 - n_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \lambda n_0 - n_1 \\ \lambda^2 n_0 - n_2 \\ \lambda^3 n_0 - n_3 \\ \lambda^4 n_0 - n_4 \end{pmatrix} = n_0 \Lambda^{(4)} - \mathbf{n}. \end{aligned}$$

Consequently, $\mathbf{P}\mathbf{C}_P = \mathbf{Q}\mathbf{A}'(n_0 \Lambda^{(4)} - \mathbf{n}) = \mathbf{0}$, since $\mathbf{n}$ is in the null space of $\mathbf{A}'$ and $\mathbf{Q}\mathbf{A}'\Lambda^{(4)} = \mathbf{0}$. Thus, $\mathbf{C}_P$ is the camera centre of $\mathbf{P}$ as required.

Finally, we wish to show that any line may be written in the form

$$\lambda(n_0, n_1, n_2, n_3)^\top - (n_1, n_2, n_3, n_4)^\top$$

for $\mathbf{n}$ the null space of some $\mathbf{A}'$ representing the coefficients of a quartic curve. First, it may easily be verified that any vector $\mathbf{n}$ can appear as the generator of the null-space of a suitable matrix $\mathbf{A}'$ representing a quartic curve, except for vectors of the form $(1, \alpha, \alpha^2, \alpha^3, \alpha^4)^\top$ for some α (including

the limit case $(0, 0, 0, 0, 1)^\top$ when $\alpha = \infty$). This exception is necessary, since if $\mathbf{A}'(1, \alpha, \alpha^2, \alpha^3, \alpha^4)^\top = \mathbf{0}$ then $\theta - \alpha$ is a common factor of the polynomials $a_i(\theta)$ where $\mathbf{A}'\Theta^{(4)} = (a_1(\theta), a_2(\theta), a_3(\theta), a_4(\theta))^\top$, and so $\mathbf{A}'$ does not represent a quartic curve (see definition 3.2). Note however that for these excluded vectors, the expression $\lambda(n_0, n_1, n_2, n_3)^\top - (n_1, n_2, n_3, n_4)^\top = (\lambda - \alpha)(1, \alpha, \alpha^2, \alpha^3)^\top$ does not represent a line as λ varies, but only a single point. Thus, we need only show that any line in space can be represented in the above form for some choice of $\mathbf{n}$. This line passes through the points $(n_0, n_1, n_2, n_3)^\top$ and $(n_1, n_2, n_3, n_4)^\top$. Consider a line passing through two arbitrary points $\mathbf{p} = (p_0, p_1, p_2, p_3)^\top$ and $\mathbf{q} = (q_0, q_1, q_2, q_3)^\top$. We need to show that there exist values a, b, c, d such that $a\mathbf{p} + b\mathbf{q} = (n_0, n_1, n_2, n_3)^\top$ and $c\mathbf{p} + d\mathbf{q} = (n_1, n_2, n_3, n_4)^\top$. This leads to a set of equations of the form

$$\begin{bmatrix} p_1 & q_1 & -p_0 & -q_0 \\ p_2 & q_2 & -p_1 & -q_1 \\ p_3 & q_3 & -p_2 & -q_2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0}$$

which must always have a non-zero solution for a, b, c, d .

Thus, this theorem has shown that both the cubic/line and rational quartic configurations given in Theorem 2.1 are critical, for arbitrary cubic/line pairs and for arbitrary rational quartics. Note that as a corollary of this result an arbitrary rational quartic will always be contained in a quadric surface (the critical quadric for this configuration).

As shown, a configuration of cameras on a line viewing a twisted cubic has a conjugate configuration. However, the ambiguity is broken by a single camera not lying on the same straight line, but not on the twisted cubic. This may be argued as follows (omitting some details). Let $\mathbf{P}$ be a further camera, not lying on the twisted cubic. Then the view of the twisted cubic as seen by the new camera is a plane cubic curve. Let $\mathbf{Q}$ be a possible conjugate camera. As seen from $\mathbf{Q}$ the conjugate quartic curve must also therefore be a cubic. It follows from this that the camera centre of $\mathbf{Q}$ must lie on the quartic curve, and hence according to Theorem 6.9 the camera centre of $\mathbf{P}$ must lie on the straight line of the other cameras.

In a similar way it may be shown that adding a single camera $\mathbf{Q}$ to the conjugate (rational quartic) configuration will break the ambiguity unless the centre of $\mathbf{Q}$ lies on the quartic curve. Filling in the details of these result would lead us too far from the main subject of this paper.

7 Classification of all multiview critical configurations.

This work aims towards a complete determination of all multiview critical configurations, following on from the

work of several other authors (including [6, 2, 3]). The basis for a classification is the result of [2] that in a critical configuration for three or more views, all the points must lie on the intersection of three ruled quadrics (not necessarily distinct). The question remains whether all possible intersections of three ruled quadrics lead to a critical configuration. It is a relatively straight-forward task to enumerate all possible ways in which three quadrics can intersect. Thus, a natural way of attacking the complete determination of multi-view critical configurations is on a case-by-case basis according to the possible intersections of three quadrics. The progress so far will be discussed below.

1. If the three quadrics meet in isolated points (generically eight points), then the configuration is critical for 7 of those points, and the three camera centres defining the quadrics ([6, 4]).
2. If the three quadrics are the same, then all the cameras in the conjugate configuration are collinear ([2]). The critical configuration given in this paper seem to be typical for this case. The critical set does not consist of the complete quadric, but only of a rational quartic curve lying on the quadric. The conjugate configuration is a twisted cubic/line configuration. The critical configuration extends to any number of views and points.
3. If the three quadrics are linearly dependent and hence define a pencil of quadrics, then the configuration is critical, as shown in [3]. The intersection of the quadrics is an elliptic quartic curve and any number of points and cameras lying on this curve constitute a critical configuration. There are infinitely many different conjugates for every critical configuration.
4. The case where the three quadrics meet in a twisted cubic is discussed in this paper, and it is shown that any number of points lying on the twisted cubic, plus any number of cameras on an arbitrary line constitute a critical configuration.
5. Other possible intersections of three quadrics involve conics and lines. The possible configurations may easily be enumerated and considered one-by-one. Preliminary results show that all these possible quadric intersections lead to critical configurations.

Thus, this paper and [3] dispose of the main possible cases of multi-view critical configurations.

8 Conclusion

This paper has identified two new families of critical configurations for three or more views. In doing so it has made

a large step towards the complete description of all critical configurations for three or more uncalibrated views.

The twisted cubic critical configuration shows an example of a shape that may not be reconstructed unambiguously from any number of views, provided the camera is moving in a straight line. However, a single image taken from a point not on a common line with the other camera centres is sufficient to break the ambiguity.

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