FAST INVERSION OF THE RADON TRANSFORM USING LOG-POLAR COORDINATES AND PARTIAL BACK-PROJECTIONS∗

FREDRIK ANDERSSON†

Abstract. In this paper a novel filtered back-projection algorithm for inversion of a discretized Radon transform is presented. It makes use of invariance properties possessed by both the Radon transform and its dual. By switching to log-polar coordinates, both operators can be expressed in a displacement invariant manner. Explicit expressions for the corresponding transfer functions are calculated. Furthermore, by dividing the back-projection into several partial back-projections, it can be performed by means of finite convolutions, and hence implemented by an FFT-algorithm. In this way, a fast and accurate reconstruction method is obtained.

Key words. Radon transform, filtered back-projection.

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1. Introduction. The two-dimensional Radon transform is the mapping from (sufficiently regular) functions on \( \mathbb{R}^2 \) to line integrals in \( \mathbb{R}^2 \),

\[
\mathcal{R} f(\theta, s) = \int_{x \cdot \theta = s} f(x) \, dx,
\]

i.e. the integral of \( f \) over the line with normal direction \( \theta \) and (signed) distance \( s \) to the origin. The standard method of inverting the Radon transform, filtered back-projection, uses the back-projecting operator \( \mathcal{R}^\# \), which integrates over all lines passing through a point. The main drawback of computer implementations of this method is that they are relatively computationally expensive. The computational complexity in computing the back-projection is of type \( O(q^3) \). Over the years, several faster (\( O(q^2 \log q) \)) reconstruction methods have been developed, but the ones of filtered back-projection type are mostly preferred in applications for reasons of quality. In this paper we make use of invariance properties of \( \mathcal{R}^\# \), for reconstruction in \( O(q^2 \log q) \) time.

It is immediate to see that the Radon transform is invariant under rotation. In addition it has the scale invariant property, for \( c > 0 \),

\[
\mathcal{R}[f(c \cdot)](\theta, s) = c \mathcal{R}[f(\cdot)](\theta, cs).
\]

The back-projecting operator is also invariant under rotation and in fact strictly scale invariant. By changing to log-polar coordinates \( (\theta, s) \mapsto (\theta, \rho) \), where \( s = e^\rho \), it is possible to express both the Radon transform (apart from scaling) and the back-projecting operator by means of convolutions. This opens for the construction of an FFT-based algorithm for fast computation of the back-projection.

However, the nonlinear coordinate change \( s \mapsto e^\rho \) makes direct usage of the convolution structure impractical in computing the total back-projection. A uniformly sampled interval in the \( \rho \)-variable containing points close to the origin in the \( s \)-variable, must be very densely sampled in order to obtain a descent sample distance with respect to \( s \), for points not close to the origin. However, this problem can be avoided through using the concept of partial back-projection introduced in §3.3.

†Centre for Mathematical Sciences, Lund University/LTH, P.O. Box 118, S-22100 Lund, Sweden. (fa@maths.lth.se).
Making use of the above invariance properties by introducing log-polar coordinates, in combination with the concept of partial back-projection, thus allows development of a fast algorithm for computing the back-projection. The quality of the reconstructions made from the algorithm presented in this paper is about the same as the one calculated in the standard setting, but in addition the method is comparable with other fast ones in speed.

2. Preliminaries. When defined on the unit cylinder $S = S^1 \times \mathbb{R}$, the two-dimensional Radon transform (defined by (1.1)) is even, i.e. $Rf(-\theta, -s) = Rf(\theta, s)$, since each line can be parameterized in two ways. Here $S^1$ denotes the unit circle, parameterized by $[-\pi, \pi)$. Sometimes it is convenient to work on $S_+ = S^1 \times \mathbb{R}_+$ or $S_{1/2} = [-\pi/2, \pi/2) \times \mathbb{R}$ instead, both sufficient to describe all lines in the plane. It is well known that the Radon transform as a mapping $S(R^2) \to S(S)$ is invertible. In this paper we are especially interested in dealing with the Radon transform of compactly supported functions, and inversion techniques for such. The Radon transform can then (possibly combined with a suitable preceding translation) be viewed as a invertible mapping $C_0^\infty(S_+) \to C_0^\infty(S_+)$.

The dual Radon transform $R^\#$ integrates functions defined on $S$ over subsets of $S$ corresponding to lines passing through a point $x \in \mathbb{R}^2$,

$$R^\# g(x) = \int_{S^1} g(\theta, x \cdot \theta) \, d\theta. \quad (2.1)$$

It is dual in the sense that

$$\int_{S} (Rf)(\theta, s) g(\theta, s) \, ds d\theta = \int_{\mathbb{R}^2} f(x)(R^\# g)(x) \, dx,$$

and is commonly in the literature referred to as the back-projecting operator.

In order to write the inversion formula for the Radon transform in a simple form, one introduces the operator $J$ acting on (sufficiently regular) functions in $S$, defined by

$$J = \frac{\partial}{\partial s} \mathcal{H},$$

where $\mathcal{H}$ denotes the Hilbert transform with respect to the second variable in $S$, i.e.

$$\mathcal{H} f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-y} f(y) \, dy,$$

where the integral is interpreted as a principal value.

An inversion formula then reads as

$$f = \frac{1}{2} \frac{1}{\sqrt{2\pi}} R^\# J R f. \quad (2.2)$$

For more details, c.f. [14]. The factor $1/2$ on the right hand side is due to the fact that in the interpretation of $R^\#$, each line is taken into account twice. Using instead $S_+$ or $S_{1/2}$, the factor $1/2$ is removed. The operator $J$ is cannot be defined directly on $S_+$, hence it is instead defined as the restriction of the action on $S$ to $S_+$.

In practice, where Radon data are given on a discrete sampling grid, the operator $J$ is usually implemented by a discrete convolution in the $s$-variable with a bandlimited filter. More specifically, one makes use of the formula

$$W * f = R^\# (w * Rf), \quad (2.3)$$
where $\ast$ denotes one dimensional convolution with respect to the second variable, $s$, in $S$, and where $W$ and $w$ are related by $W = R^#w$, c.f. [14]. By choosing $W$ approximating a $\delta$-distribution, we obtain an approximate reconstruction. More specifically, one usually chooses $W = W_b$ to be radially symmetric and band-limited with some cut-off frequency $b$. When chosen in this way, $w = w_b$ depends only on the $s$-variable, in which it is band-limited by $b$, c.f. [14].

By approximating the continuous convolution in 2.3 by a discrete one, the filtering step $w_1 \ast R f$ can be accomplished, by use of FFT, with a time complexity $\mathcal{O}(q^2 \log q)$, if the number of parallel lines and directions both are $q$. A more time-consuming step is the computation of the back-projection. The straightforward numerical implementation of $R^#$ uses, for each discrete direction sample point $\theta_j$, some kind of interpolation in the $s$-variable to approximate $g(\theta_j, x \cdot \theta_j)$ in (2.1), in combination with some quadrature rule on $S^1$:

$$R^#_{\theta_j} g(x) = \sum_j \alpha_j g(\theta_j, x \cdot \theta_j).$$

Here, roughly $q$ values are summed up at each reconstruction point, giving a time complexity of order $\mathcal{O}(q^3)$ for $q^2$ reconstruction points. Reconstruction methods which are based on (2.2) are referred to as filtered back-projection algorithms.

Several suggestions on how to invert the Radon transform in $\mathcal{O}(q^2 \log q)$ time has appeared in the literature during the years. Most common are Fourier based methods, making use of the "Fourier slice theorem":

$$\mathcal{F}_s(\mathcal{R} f)(\theta, \sigma) = \mathcal{F}_2(f)(\sigma \theta),$$

where $\mathcal{F}_s$ denotes the one-dimensional Fourier transform with respect to $s$ on $S$, and $\mathcal{F}_2$ denotes the two-dimensional cartesian Fourier transform on $\mathbb{R}^2$. Using this with natural uniform discretizations of the $\theta$- and $\sigma$-variables results in knowledge of the two-dimensional Fourier transform on the right hand side on a non-uniform grid. Direct interpolation onto a rectangular grid followed by FFT inversion result in $\mathcal{O}(q^3 \log q)$ complexity, but gives rise to unacceptable artifacts. To cope with this one can e.g. use over-sampling combined with more sophisticated interpolation, as suggested in [14], or use fast Fourier algorithms for unequally spaced data, c.f. [2], [16].

Fast techniques for filtered back-projection algorithms are presented in [15], [5], [3], [4], where the back-projection is calculated recursively in $\mathcal{O}(q^2 \log q)$ time.

In this paper we present a back-projection algorithm which uses a log-polar grid for treating the back-projection part. Log-polar grids have previously been used in an ART fashion, cf [8], to derive a system of linear equations, which after regularization can be treated with standard numerical techniques. However, according to [8], this approach was not successful in competition with the standard filtered back-projection algorithm, neither in speed nor quality.

To begin with we give a discussion about some properties of the continuous Radon transformation and its dual, which are both of interest on its own, and useful for the discrete approximative inversion presented below. A somewhat reminiscent discussion is given in [1], where some closed form formulas, involving Tchebychev polynomials, are given starting from a polar representation. A numerical implementation is also presented, in [1], but the number of computations needed are of order $\mathcal{O}(q^3)$. 

Fast Inversion of the Radon Transform using Log-polar Coordinates

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3.1. Convolution operators. Consider the cylinder $S_+ = S^1 \times \mathbb{R}_+$ as a group, provided with the algebraic structure inherited from its components, i.e. the additive group $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ (addition modulo $2\pi$), and the multiplicative group $\mathbb{R}_+$ (positive real numbers).

Let $z = (\theta, s) \in S_+$. The group operation on $S_+$, written multiplicatively, is

$$(\theta_1, s_1)(\theta_2, s_2) = ((\theta_1 + \theta_2) \mod 2\pi, s_1 s_2) \ .$$

The Haar measure on $S_+ = S^1 \times \mathbb{R}_+$ is inherited from the components, and can be written $dh(z) = d\theta ds/s$. Hence

$$\int_{S_+} f(z) \, dh(z) = \int_0^{2\pi} \int_0^\infty f(\theta, s) \, d\theta \, ds/s \quad \text{for} \quad f \in \dot{C}_0^\infty(S_+) \ ,$$

where $\dot{C}_0^\infty(S_+)$ is the $C_0^\infty$ class on $S_+$, with support outside the origin $S^1 \times \{0\}$. The Haar property means that

$$\int_{S_+} f(wz) \, dh(z) = \int_{S_+} f(z) \, dh(z) \quad \text{for} \quad w \in S_+, \ f \in \dot{C}_0^\infty(S_+) \ .$$

There exists a natural isomorphism between $S_+$ and the punctured complex plane $\dot{\mathbb{C}} = \mathbb{C} \setminus \{0\}$ considered multiplicatively, parameterized by $(\theta, s) \mapsto se^{i\theta}$. Using the cartesian representation $z = x + iy$ for $\dot{\mathbb{C}}$, the Haar measure on $S_+$ can be written

$$dh(z) = d\theta ds = \frac{dx \, dy}{x^2 + y^2} \ .$$

If $\dot{\mathbb{C}}$ is represented instead by coordinates $se^{i\theta} = e^\rho e^{i\theta}, \rho \in \mathbb{R}, \tag{3.1}$ which we will refer to as log-polar coordinates, then the Haar measure on $S_+$ becomes

$$dh(z) = d\theta dp.$$

Let $\lambda$ be the distribution that represents integration over the line $x = 1$ in $\dot{\mathbb{C}}$,

$$\lambda : f \mapsto \int_{-\infty}^{\infty} f(1, y) \, dy = \oint_{\mathbb{C}} f(x, y) \delta(x - 1) \, dx \, dy \ , \ f \in \mathcal{S}(S_+) \ .$$

In the $(\theta, s) = z$ representation, it can be written

$$\lambda : f \mapsto \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta, \frac{1}{\cos^2 \theta}) \frac{d\theta}{\cos^2 \theta} = \int_{S_+} f(\theta, s) \delta(s \cos \theta - 1) s \, ds \, d\theta = \int_{S_+} f(z) \lambda(z) \, dh(z) \ , \ \text{with} \ \lambda(z) = s^2 \delta(s \cos \theta - 1) \ .$$

Let lines in $\dot{\mathbb{C}}$ be denoted $L_\xi$, where $\xi$ is the footprint of the normal of the line through the origin. The Radon transform (1.1) can be expressed as

$$g(\xi) = Rf(\xi) = |\xi| \int_{S_+} f(z\xi) \lambda(z) \, dh(z) \ . \tag{3.2}$$
This follows from the fact that in the last integral, the distribution \( \lambda \) is applied to a function that is obtained from \( f \) by a similarity transformation of the coordinate plane, such that the line \( L_\xi \) is transferred to \( L_1 \), which is the support of \( \lambda \). Using \( |\xi| \) to compensate for the change in scale, we obtain the Radon transform.

The duality between points on a line in \( \mathbb{C} \) and lines through a point in \( \mathbb{C} \) motivates interest for the distribution \( \lambda^\#(z) = \lambda(1/z) \), formally defined by

\[
\zeta : f \mapsto \int_{S_+} f(1/z) \lambda(z) \, dh(z),
\]

\[
\zeta^\# : f \mapsto \int_{S_+} f(1/z) \lambda^\#(z) \, dh(z), \quad f \in \mathcal{C}_0^\infty(S_+).
\]

which, if \( f \) is interpreted as a function on lines in \( \mathbb{C} \), is the integral over all lines through the point \( 1 \in \mathbb{C} \). Also note that

\[
\int_\alpha^\beta \int_{\mathbb{R}^+} f(s, \theta) \lambda^\#(s, \theta) \, ds \, d\theta
\]

is the integral over all lines through the point \( 1 \in \mathbb{C} \) with normal direction in the interval \( [\alpha, \beta] \).

The distributions \( \zeta(z) = \lambda(1/z) \) and \( \zeta^\#(z) = \lambda^\#(1/z) \), formally defined by

\[
\zeta : f \mapsto \int_{S_+} f(1/z) \lambda(z) \, dh(z),
\]

\[
\zeta^\# : f \mapsto \int_{S_+} f(1/z) \lambda^\#(z) \, dh(z), \quad f \in \mathcal{C}_0^\infty(S_+).
\]

will be crucial in what follows. In the \((\theta, s) = z\) representation, these can explicitly be written

\[
\zeta(z) = s^{-2} \delta(s^{-1} \cos \theta - 1),
\]

\[
\zeta^\#(z) = \delta(s \cos \theta - 1).
\]

Using the Haar property on the integral, the formula (3.2) can also be written

\[
\mathcal{Z} f(\xi) = \frac{\mathcal{R} f(\xi)}{|\xi|} = \int f(z) \zeta(\xi z) \, dh(z).
\]

Apart from the scaling factor \( |\xi| \), the Radon transform thus can be expressed as a convolution on \( \mathcal{C}_0^\infty(S_+) \).

For the back-projecting operator (2.1), note that if \( \xi \in \mathbb{C} \) and if \( g \in \mathcal{C}_0^\infty(S_+) \) is interpreted as a function on lines in \( \mathbb{C} \), then

\[
\mathcal{R}^\# g(\xi) = \int_{S_+} g(\xi z) \lambda^\#(z) \, dh(z),
\]

is the integration over all lines that passes through \( \xi \). Hence, by using the Haar property again,

\[
\mathcal{R}^\# g(\xi) = \int_{S_+} g(z) \zeta^\#(\xi z) \, dh(z),
\]

(3.5)
or in the log-polar representation
\[ \mathcal{R}^\# g(\rho, \theta) = \int_{S^+} g(\rho', \theta') \zeta^\#(\rho - \rho', \theta - \theta') \, d\rho \, d\theta. \] (3.6)

This fact opens a possibility to perform the inversion in (2.2) by means of an appropriate Fourier transform.

### 3.2. Fourier analysis.

The Fourier transform \( \mathcal{F}_{S^1} = \mathcal{F} \) on \( S_+ \) is a compound of the Fourier series transform on \( S^1 \) and the Mellin transform on \( \mathbb{R}^+ \):
\[
\mathcal{F} : f(\theta, s) \mapsto g(\mu, \sigma) = \int_0^{2\pi} \int_0^{\infty} e^{-i\mu \theta} s^{-\sigma} f(\theta, r) \, \frac{ds}{s} \, d\theta, \quad \mu \in \mathbb{Z}, \, \sigma \in \mathbb{C}.
\]

In log-polar representation (3.1) the Fourier transform instead becomes a compound of the Fourier series transform and the Laplace transform on \( \mathbb{R}^+ \),
\[
\mathcal{F} : f(\theta, \rho) \mapsto g(\mu, \sigma) = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{-i\mu \theta} e^{-\sigma \rho} f(\theta, r) \, d\rho \, d\theta, \quad \mu \in \mathbb{Z}, \, \sigma \in \mathbb{C}.
\]

As the operators \( \mathcal{Z} \) in (3.5) and \( \mathcal{R}^\# \) can be expressed as convolutions on \( S^+ \), it suffices to calculate the corresponding transfer functions for determination of \( \mathcal{F}_\zeta \) and \( \mathcal{F}_{\zeta^\#} \).

It is readily verified using (3.2) that for \( \xi = r e^{i\psi} \),
\[
\mathcal{Z} f(\xi) = \int_{S^1} f(\theta + \psi, rs) \delta(s \cos \theta - 1) s \, ds \, d\theta = \int_{-\pi/2}^{\pi/2} f(\theta + \psi, r \cos \theta) \frac{1}{\cos^2 \theta} \, d\theta.
\]

Let \( f(\theta, s) = s^\sigma e^{i\mu \theta} \). Then the transfer function for \( \mathcal{Z} \) is obtained by
\[
\mathcal{Z} f(\xi) = \int_{-\pi/2}^{\pi/2} e^{i\mu(\theta + \psi)} \frac{1}{\cos^2 \theta} \, d\theta = r^\sigma e^{i\mu \psi} \int_{-\pi/2}^{\pi/2} e^{i\mu \theta} \cos^2 \theta \, d\theta = f(\xi) \mathcal{F}_{\zeta}(\mu, \sigma).
\]

Here, \( \mathcal{F}_{\zeta}(\mu, \sigma) \) converges in the classical sense for \( \text{Re} \sigma < -1 \), where it defines an analytic function.

Similarly, the transfer function for the back-projecting operator \( \mathcal{R}^\# \) is obtained by
\[
\mathcal{R}^\# f(\xi) = \int_{-\pi/2}^{\pi/2} e^{i\mu \theta} (r \cos(\theta - \psi))^\sigma \, d\theta = \int_{-\pi/2}^{\pi/2} e^{i\mu(\theta + \psi)} (r \cos \theta)^\sigma \, d\theta = r^\sigma e^{i\mu \psi} \int_{-\pi/2}^{\pi/2} e^{i\mu \theta} \cos^\sigma \theta \, d\theta = f(\xi) \mathcal{F}_{\zeta^\#}(\mu, \sigma),
\]

and \( \mathcal{F}_{\zeta^\#}(\mu, \sigma) \) converges in the classical sense for \( \text{Re} \sigma > -1 \).

Note that \( \mathcal{F}_{\zeta}(\mu, \sigma) \) and \( \mathcal{F}_{\zeta^\#}(\mu, \sigma) \) are not simultaneously (classically) well defined, and furthermore the resemblance between the two Fourier transforms:
\[
\mathcal{F}_{\zeta}(\mu, \sigma) = \mathcal{F}_{\zeta^\#}(\mu, -\sigma - 2).
\]

In order to derive explicit expressions for the Fourier transforms above, we prove the following proposition:
Proposition 3.1. Let \( \mu \in \mathbb{Z}, \alpha > -1 \), and
\[
P(\mu, \alpha) = \int_{-\pi/2}^{\pi/2} e^{i\mu \theta} \cos^\alpha(\theta) \, d\theta.
\] (3.7)

Then \( P(\mu, \alpha) \) is an analytic function given by
\[
P(\mu, \alpha) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2} + 1\right)\Gamma\left(\frac{\alpha+2}{2} + 1\right)}.
\]

Proof. By two partial integrations it follows that
\[
-\mu^2 P(\mu, \alpha) = \int_{-\pi/2}^{\pi/2} \frac{d^2}{d\theta^2} (e^{i\mu \theta}) (\cos^\alpha(\theta))^\alpha \, d\theta = \int_{-\pi/2}^{\pi/2} e^{i\mu \theta} (\alpha (\alpha - 1) (\cos \theta)^{\alpha - 2} - \alpha^2 (\cos \theta)^{\alpha}) \, d\theta = \alpha (\alpha - 1) P(\mu, \alpha - 2) - \alpha^2 P(\mu, \alpha),
\]
giving rise to the following recursion formula:
\[
P(\mu, \alpha - 2) = \frac{\alpha}{\alpha - 1} (1 - \frac{\mu^2}{\alpha^2}) P(\mu, \alpha).
\]

Hence, for all positive integers \( n \),
\[
P(\mu, \alpha - 2) = \left( \prod_{k=0}^{n} \frac{\alpha + 2k}{\alpha + 2k - 1} \right) \left( \prod_{k=0}^{n} \left( 1 - \frac{\mu^2}{(\alpha + 2k)^2} \right) \right) P(\mu, \alpha + n) = \frac{2^n \Gamma\left(\frac{\alpha}{2} + n + 1\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{\Gamma\left(\frac{\alpha + 1}{2} + n\right)}{2^n \Gamma\left(\frac{\alpha + 1}{2} + n\right)} \left( \prod_{k=0}^{n} \left( 1 - \frac{\mu^2}{(\alpha + 2k)^2} \right) \right) P(\mu, \alpha + 2n) = \frac{\mathcal{B}(\alpha - \frac{1}{2}, \frac{1}{2})}{\mathcal{B}(n + \frac{\alpha}{2} + \frac{1}{2})} \left( \prod_{k=0}^{n} \left( 1 - \frac{\mu^2}{(\alpha + 2k)^2} \right) \right) P(\mu, \alpha + 2n),
\] (3.8)

where \( \mathcal{B} \) denotes the beta function and where the identity
\[
\mathcal{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n + 1)},
\]
has been used. Recall that the beta function is defined by
\[
\mathcal{B}(m + 1, n + 1) = 2 \int_{0}^{\pi/2} (\cos \theta)^{2m+1} (\sin \theta)^{2n+1} \, d\theta.
\] (3.9)

Introduce
\[
h_n(t) = \frac{(\cos t)^n}{\mathcal{B}(\frac{n}{2} + \frac{1}{2})},
\]
which by using 3.9 easily is shown to satisfy,
\[
\lim_{n \to \infty} h_n(t) = \delta(t),
\]
for $|t| < \pi$. Using this in combination with (3.7), we rewrite (3.8),

$$P(\mu, \alpha - 2) =$$

$$= \lim_{n \to \infty} B\left(\frac{\alpha}{2} - \frac{1}{2}, \frac{1}{2}\right) \left(\prod_{k=0}^{n} \left(1 - \frac{\mu^2}{(\alpha + 2k)^2}\right)\right) \int_{-\pi/2}^{\pi/2} e^{i\mu\theta} \frac{(\cos \theta)^{\alpha + 2n}}{B(n + \frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2})} d\theta =$$

$$= B\left(\frac{\alpha}{2} - \frac{1}{2}, \frac{1}{2}\right) \left(\prod_{k=0}^{\infty} \left(1 - \frac{\mu^2}{(\alpha + 2k)^2}\right)\right). \quad (3.10)$$

From the identity

$$\frac{\Gamma^2(n + 1)}{\Gamma(n + ix + 1)\Gamma(n - ix + 1)} = \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{(n + k)^2}\right),$$

on page 699 in [20], cf. formula (64), it follows that

$$P(\mu, \alpha) = B\left(\frac{\alpha + 1}{2}, \frac{1}{2}\right) \left(\prod_{k=1}^{\infty} \left(1 - \frac{\mu^2}{(\alpha - 2k)^2}\right)\right) = \frac{\Gamma(\frac{\alpha + 1}{2})\Gamma(\frac{\alpha + 2}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha + \mu}{2} + 1)\Gamma(\frac{\alpha - \mu}{2} + 1)}. \quad (3.11)$$

By extending $P(\mu, \alpha)$ of proposition 3.1 analytically, we conclude that

$$F_\zeta(\mu, \sigma) = \frac{\Gamma\left(-\frac{\sigma - 1}{2}\right)\Gamma\left(-\frac{\sigma}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{\sigma + \mu}{2}\right)\Gamma\left(-\frac{\sigma - \mu}{2}\right)},$$

$$F_\zeta^\#(\mu, \sigma) = \frac{\Gamma\left(\frac{\sigma + 1}{2}\right)\Gamma\left(\frac{\sigma + 2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\sigma + \mu}{2} + 1\right)\Gamma\left(\frac{\sigma - \mu}{2} + 1\right)}. \quad (3.11)$$

Below, figure 3.1 illustrates properties of the function $F_\zeta^\#(\mu, i\omega)$. Its energy content

![Fig. 3.1. The absolute value and real part of $F_\zeta^\#(\mu, i\omega)$](image)

is mainly contained in a neighborhood of the origin and along the line $\mu = 0$. Note that although its absolute value it rather smooth, the real and imaginary parts are strongly oscillating.
3.3. Partial back-projection. Our aim is to present a fast procedure for computing the back-projection as a discrete convolution on a uniformly sampled grid in log-polar coordinates. However, the counterpart in polar coordinates will be non-uniform, corresponding to a dense assembling of grid points close to the origin in C. This assembling causes large variation in the density of both reconstruction grid points and data sample points. By moving the origin it is possible to obtain a more uniform structure within the grid points of the reconstruction region, but this does not simplify the treatment of the data sample points. The difficulty here is the need of dealing with lines with various distance to the origin, and this problem consist independently of where the origin is situated, if lines with all directions must be considered.

To cope with this, we divide the measurement data into \( m \geq 3 \) (disjoint) sets, each containing lines with directions in an interval of length \( \frac{\pi}{m} \). Corresponding to each such set, we choose an origin outside the region of interest, in such a way that the back-projection from lines with directions within each interval, cf. (3.3), can be calculated by means of finite integrals. We refer to this procedure as partial back-projection. By putting these partial back-projections together it is possible to obtain a total back-projection. To begin with, we consider one special case.

Let \( \beta = \frac{\pi}{m} \) for some positive integer \( m \geq 3 \), and let \( a_R \) denote the radius of the largest circle inscribed in a sector with unit radius and central angle \( \beta \), cf. figure 3.2.

![Fig. 3.2. Lines with directions \( \theta \in [-\frac{\beta}{2}, \frac{\beta}{2}] \), passing through a circle inscribed in a segment (3.2). It is straightforward to show that](image)

\[
a_R = \frac{\sin(\frac{\beta}{2})}{1 + \sin(\frac{\beta}{2})}, \tag{3.12}
\]

Consider the set of all lines, \( L_{\theta,s} \) with normal directions \( |\theta| \leq \frac{\beta}{2} \), with respect to the symmetry axis of the sector, passing through the inscribed circle, cf. figure 3.2. It is clear that the normal distances to the origin, \( s \), of these lines will be in the interval \((a_m, 1)\), where

\[
a_m + a_R = (1 - a_R) \cos(\frac{\beta}{2}), \tag{3.13}
\]

or by using 3.12

\[
a_m = \frac{\cos(\frac{\beta}{2}) - \sin(\frac{\beta}{2})}{1 + \sin(\frac{\beta}{2})}. \tag{3.14}
\]
Suppose that \( f \in \mathcal{C}_0^\infty \) has its support inside the inscribed circle, and that \( \mathcal{R}f(\theta, s) \) is known. Let

\[
h^0(\theta, s) = \chi_\beta \mathcal{R}f(\theta, s),
\]

where \( \chi_\beta \) is the characteristic function of the set

\[
\{ \theta : \frac{-\beta}{2} \leq \theta < \frac{\beta}{2} \}.
\]

The contribution

\[
f_0(x) = \int_{\frac{-\beta}{2}}^{\frac{\beta}{2}} \mathcal{R}Jf(\theta, \theta \cdot x) d\theta
\]

from the back-projection of lines with directions in the interval \((\frac{-\beta}{2}, \frac{\beta}{2})\) in log-polar coordinates, for \( x \) inside the inscribed circle, may by use of (3.6) and (3.3) be written as

\[
f_0(\theta, \rho) = \int_{\frac{-\beta}{2}}^{\frac{\beta}{2}} \int_{\ln(a_m)}^{0} h^0(\theta', \rho') \zeta^\#(\theta - \theta', \rho - \rho') d\rho' d\theta'.
\]

Furthermore, if \( \zeta^\#(\theta, \rho) \) is defined as the periodic extension of \( \zeta^\#(\theta, \rho) \) for \( \theta \in [\beta, \beta] \) and \( \rho \in [\ln(a_m), 0] \), then

\[
f_0(\theta, \rho) = \int_{\frac{-\beta}{2}}^{\frac{\beta}{2}} \int_{\ln(a_m)}^{0} h^0(\theta', \rho') \zeta_p^\#(\theta - \theta', \rho - \rho') d\rho' d\theta',
\]

(3.15)

when \( \theta \in [-\frac{\beta}{2}, \frac{\beta}{2}] \) and \( \rho \in [1 - 2a_R, 1] \). This is due to the fact that the integral length in the \( \theta' \)-direction is twice the \( \theta' \)-support of \( h^0(\theta', \rho') \), and to the fact that the values of \( h^0(\theta', \rho') \) for \( \rho' \) outside \( [\ln(a_m), 0] \), is of no importance in (partial) back-projecting for \( x = (\theta, \rho) \) inside the inscribed circle. Thus, replacing \( h^0(\theta', \rho') \) with a periodic extension of \( h^0(\theta', \rho') \) as above, which is equivalent to extending \( \zeta^\# \), does not influence the result. Hence, the partial back-projection at points of interest can be calculated as a periodic convolution.

Now, let \( f \in \mathcal{C}_0^\infty(\Omega^2) \), where \( \Omega^2 \) denotes the unit disc in \( \mathbb{R}^2 \), and suppose that \( \mathcal{R}f(\theta, s) \) is known. By dividing the data into \( m \) different parts, each spanning an angle interval of \( \beta \), and to each such set make a suitable change of coordinates, one transforms the full back-projection problem to \( m \) subproblems of the form above. An illustration of the procedure, for \( m = 3 \), is given in figure 3.3. The coordinate transformation \( x \rightarrow x_\nu \), consists of, after a rescaling with \( a_R \), a rotation by \( \beta \) degrees followed a translation of the origin \( O \) to \( O_\nu \). Each part is then of the form discussed above, and adding the respective partial back-projections gives the total back-projection, since each partial back-projection integrates over disjunct intervals \((-\beta/2 + \nu\beta, \beta/2 + \nu\beta)\), which together covers an interval length of \( m\beta = \pi \). For the sake of completeness, we include the details.

Let \( \nu = 0, \ldots, m - 1 \), and introduce for each \( \nu \) new coordinates

\[
x_\nu = a_R \begin{pmatrix} \cos(\nu\beta) & \sin(\nu\beta) \\ -\sin(\nu\beta) & \cos(\nu\beta) \end{pmatrix} x + \begin{pmatrix} 1 - a_R \\ 0 \end{pmatrix},
\]
and let \( f^\nu(x_\nu) = f(x) \). Each \( f^\nu \) then has its support inside a circle inscribed in a sector with unit radius and central angle \( \beta \), as in figure 3.2. Let

\[
h^\nu = \chi_\beta \mathcal{J} \mathcal{R} f^\nu.
\]

The following relation, easily verified,

\[
x_\nu \cdot \theta = x \cdot a_R(\theta + \nu \beta) + (1 - a_R) \cos(\theta),
\]

will be useful. Any line in the \( x \)-coordinate system may be written

\[
L_{\theta + \nu \beta, s} = \{ x | x \cdot (\theta + \nu \beta) = s \},
\]

for some \( \nu \in \{0, \ldots, m - 1\} \) and \( -\frac{\beta}{2} \leq \theta < \frac{\beta}{2} \). In the \( x_\nu \)-coordinate system the corresponding line may then, by using (3.16), be written

\[
L_{\theta, a_R s + (1 - a_R) \cos(\theta)}^\nu = \{ x_\nu | x_\nu \cdot \theta = a_R s + (1 - a_R) \cos(\theta) \}.
\]

Hence,

\[
h^\nu(\theta, s) = \chi_\beta \mathcal{J} \mathcal{R} f(\theta + \nu \beta, \frac{s - (1 - a_R) \cos(\theta)}{a_R})
\]

The total back-projection for some point \( x \) inside the support of \( f \) can now, again by using (3.16), be written as

\[
\mathcal{R}^\# \mathcal{J} \mathcal{R} f(x) = \int_{S^1} \mathcal{J} \mathcal{R} f(\theta, x \cdot \theta) d\theta =
\]

\[
= 2 \sum_{\nu=0}^{m-1} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \mathcal{J} \mathcal{R} f(\theta + \nu \beta, (\theta + \nu \beta) \cdot x) d\theta =
\]

\[
= 2 \sum_{\nu=0}^{m-1} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} h^\nu(\theta, a_R(\theta + \nu \beta) \cdot x + (1 - a_R) \cos(\theta)) d\theta =
\]

\[
= 2 \sum_{\nu=0}^{m-1} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} h^\nu(\theta, x_\nu \cdot \theta) d\theta,
\]
where the factor 2 in the last expressions is due to the fact that the total integration is only performed on half of $S^1$.

We now have reduced the full back-projection problem to making $m$ partial reconstructions of the form discussed above. For future references, let us introduce

$$f_r^{\nu}(x, \nu) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h_r^{\nu}(\theta, x, \nu, \theta) d\theta.$$  

(3.18)

The inversion formula (2.2) then allows us to write

$$f(x) = \sum_{\nu=0}^{m-1} f_r^{\nu}(x, \nu).$$  

(3.19)

We’ve deduced that it is possible to compute the total back-projection as a sum of partial back-projection. The sharp cutoff caused by $\chi_\beta$ will however in practice give rise to artifacts in form of sharp lines with normal directions corresponding to the cutoff angles. Therefore it is desirable to make a smoother cutoff. This require that the supports in $\theta$-direction for the respective $h^{\nu}$ overlap. The angle $\beta$ is then not equal to $\frac{\pi}{m}$, where $m$ is the number of partial back-projections to be used, but larger. Suppose $\frac{\pi}{m} < \beta < \frac{2\pi}{m}$. By letting

$$\eta(t) = \begin{cases} e^{-\frac{|t|^2}{\beta^2}}, & \text{if } |t| < \frac{\beta}{2}; \\ 0, & \text{otherwise}, \end{cases}$$

a smooth cutoff function

$$\chi_{\beta,m}(\theta) = \frac{\eta(\theta)}{\eta(\theta) + \eta(\frac{\pi}{m} - \arccos(\cos \theta))}$$  

(3.20)

with support in $[-\frac{\beta}{2}, \frac{\beta}{2}]$, can be formed. Since

$$\sum_{k=-\infty}^{\infty} \chi_{\beta,m}(\theta + k \frac{\pi}{m}) = 1,$$

it follows that the reconstruction (3.19) is still valid if $\chi_\beta$ in (3.17) is replaced by $\chi_{\beta,m}$.

4. Discrete log-polar reconstruction.

4.1. Principles. In this section we present a description on how to use the method of section 3.3 for making reconstructions from discrete measurements. For simplicity we deal with the case of parallel beam data. Hopefully, other types of data will be treated in a sequel to this paper. Let $f \in \mathcal{C}_0^\infty(\Omega^2)$, let $g = \mathcal{R}f$ be sampled at $(\theta_j, s_l)$, $j = 0, \ldots, p-1$, $l = -q, \ldots, q$ (parallel beam geometry), where $\theta_j \in S^1$ and $s_l = \frac{l}{q}$, and let $h = \omega_n^d g$, in accordance to (2.3). The latter quantity is approximated by the discrete convolution

$$w_b d \ast g(\theta_j, s_l) = \frac{1}{q} \sum_{l=-q}^{q} w_b(s_l - s_l)g(\theta_j, s_l).$$  

(4.1)
What now remains for reconstruction is the back-projection. To this end, let the integer \( m \) be the number of partial reconstruction to be used, let \( \beta = \pi/m_{\beta} \), suppose that \( m \) and \( m_{\beta} \) divides \( p \), and let \( \theta_j = [-\beta, \frac{\beta}{2}] \).

For the construction of the discrete back-projecting operator we will make use of the interpolation results discussed in \( \S A \). To begin with, suppose \( h(\theta, s) \) is known, and sample \( h^\nu \) defined in (3.17), with \( \chi_{\beta} \) replaced by \( \chi_{\beta, m} \) from (3.20), uniformly in a log-polar representation \((\theta_j, \rho_i)\) covering \((-\frac{\beta}{2}, \frac{\beta}{2}) \times (-\ln(a_m), 0)\). Let \( I_{\Delta_2} = I_{(\Delta_\theta, \Delta \rho)} \) be a direct quasi-interpolator of order \( \beta \), with kernel \( \varphi(\theta, \rho) = \varphi_\theta(\theta)\varphi_\rho(\rho) \), where \( \Delta \theta \) and \( \Delta \rho \) is the grid spacing of \((\theta_j, \rho_i)\).

Construct

\[
\hat{h}_\nu^\zeta(\theta, \rho) = \sum_j \sum_i h^\nu[j, i] \varphi \left( \frac{\theta - \theta_j}{\Delta \theta}, \frac{\rho - \rho_i}{\Delta \rho} \right),
\]

as an approximation of \( h^\nu \). Now the back-projection of \( h^\nu \), for \( x_\nu \) represented by \( \theta \in [-\frac{\beta}{2}, \frac{\beta}{2}] \) and \( \rho \in [1 - 2a_\rho, 1] \), can be written

\[
\mathcal{R}_\nu^\# h^\nu(\theta, \rho) = \int_{-\beta}^{\beta} \int_{\ln(a_m)}^{0} h^\nu(\theta - \theta', \rho - \rho') \zeta^\#(\theta', \rho') \, d\rho' \, d\theta' =
\]

\[
\int_{-\beta}^{\beta} \int_{\ln(a_m)}^{0} \left( \sum_j \sum_i h^\nu[j, i] \varphi \left( \frac{\theta - \theta_j - \theta', \rho - \rho_i - \rho'}{\Delta \theta}, \frac{\rho - \rho_i - \rho'}{\Delta \rho} \right) \right) \zeta^\#(\theta', \rho') \, d\rho' \, d\theta' =
\]

\[
\sum_j \sum_i h^\nu[j, i] Z(\theta - \theta_j, \rho - \rho_i),
\]

(4.2)

where

\[
Z(\theta, \rho) = \int_{-\beta}^{\beta} \int_{\ln(a_m)}^{0} \varphi \left( \frac{\theta - \theta'}{\Delta \theta}, \frac{\rho - \rho'}{\Delta \rho} \right) \zeta^\#(\theta', \rho') \, d\theta' \, d\rho'.
\]

(4.3)

This follows by exchanging the order between sums and integrals. Note that \( Z \) is independent of the Radon data.

Due to this structure, it is particularly convenient to make reconstructions on some uniformly sampled \((\theta, \rho)\)-grid, e.g. the same upon which \( h^\nu \) was resampled, as this enables computation of the discrete convolutions by means of two-dimensional FFT (after appropriate zero padding). Once the partial back-projections \( f^\nu_\nu \) are computed on the \((\theta_j, \rho_i)\)-grid, it remains only to interpolate them onto a cartesian representation and add the results to obtain a reconstruction \( f \). A survey of relevant interpolation methods is given in \( \S A \).

The procedure discussed above involves three quasi-interpolators. The first one, which was not discussed above, is needed in the resampling onto the uniform \((\theta_j, \rho_i)\)-grid. Let it be denoted by \( I_{\Delta_1} = I_{\frac{\beta}{\pi \cdot 2}} \), and its interpolating order by \( a_1 \). Hence, \( h[j, i] \) above should be replaced by \( (I_{\Delta_1} h)[j, i] \). The second one, represented by \( I_{\Delta_2} \) above, is naturally incorporated into \( Z(\theta, \rho) \) as described by 4.3, and the third one, \( I_{\Delta_3} \), of interpolating order \( a_3 \), is needed when adding up the parts. In short, a pseudo code for the reconstruction is presented below:

**Algorithm 4.1.**

```matlab
function f=iradonlp(g,m,mbeta);
[p,q]=size(g);
h=wfilter(g);
```
Fredrik Andersson

\[ \text{FZ=fft2(get\_Z(p,q,mbeta));} \]
\[ \text{chi=get\_chi(m,mbeta,q);} \]
\[ f=0; \]
\[ \text{for nu=0:m-1,} \]
\[ \text{hlp=interp\_pol2lp(chi.*h(nu*p/m+(0:p/mbeta-1),:));} \]
\[ \text{hlp=[hlp;zeros(size(hlp))];} \]
\[ \text{flp=ifft2(FZ*fft2(hlp));} \]
\[ f=f+interp\_lp2cart(flp); \]
\[ \text{end;} \]
\[ \text{end;} \]

Let us discuss the first interpolation step slightly more into detail. Since, for each fixed \( \theta \), \( h(\theta,s) \) is band-limited by \( b \), the first interpolation step can actually be computed exactly from \( h(\theta_j, s_l) \) by (A.1), as long as the Nyquist condition \( b < \pi q \) is satisfied. However, this is quite time consuming, and ruins the time gain achieved by using log-polar coordinates in the back-projection.

The non-uniformity between data in polar and log-polar representation respectively, requires use of more sample points in the log-polar representation in order to avoid too much loss of information. Suppose \( \Delta \rho = -\ln(a_m/q)q' - 1 \), where \( q' = \kappa(2q + 1) \) for some over sample factor \( \kappa \). If the filter band-width of (4.1) is given by \( b = \pi q \), then by choosing

\[ \kappa > \frac{-\ln(a_m)}{2a_R} \]

the knowledge of \( h(\theta, \rho_i) \) at all grid points suffice to reconstruct \( h(\theta, \rho_i) \), cf. [21]. A typical choice here is \( \kappa = 2 \).

Remark. The combination of uniformly spaced FFT and the interpolation scheme described above is in principle the same as in the procedures used in fast Fourier transforms for unequally spaced data, cf. [2]. Usage of the fast implementations available for such routines allows simple and fast implementation of the algorithm described above. It should be stressed that although the tools used are the same as in e.g. [16], the underlying method is quite different; the one of this paper is based on the filtered back-projection technique, whereas the others has been based on the Fourier slice theorem.

4.2. Error analysis. In the algorithm described above, errors are introduced at several places. The first one is made in the filtering step, caused by the discrete convolution (4.1). This is common for algorithms of filtered back-projection type, and estimated in [14] by

\[ |w_b * g - w_b * g|_\theta(s) \leq \frac{1}{2} \int_{\sigma \geq b} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \sigma = e_1. \] (4.4)

To describe the ones caused by interpolation we need to introduce modified Sobolev norms, for polar and log-polar coordinates, respectively. The main reason for not using the natural definitions is that since each filtering \( h \) of Radon data of some compactly supported function is not compactly supported, and especially since the filtering does not vanish in a neighborhood of \( s = 0 \), the natural Sobolev norm in log-polar coordinates of \( h \) is in general, if it exists, infinite.

Similarly to the definition (A.6), let

\[ ||f||_{H^\gamma_{\text{pol}}} = \sum_{|\alpha| \leq \gamma} \int \int |D^\alpha_{\theta,s} f(\theta,s)|^2 \chi_{\text{pol}}(\theta,\rho) d\theta d\sigma, \]

where
and
\[ \|f\|_{H^\gamma_{\rho}} = \sum_{|\alpha| \leq \gamma} \int \int |D^\alpha \rho f(\theta, \rho)|^2 \chi_{\rho}(\theta, \rho) \, d\theta \, d\rho, \]
be polar and log-polar Sobolev norms, respectively. Here \( f(\theta, s) \) and \( f(\theta, \rho) \) are connected through the change of variables
\[ \rho = \ln(\frac{s - (1 - a_R) \cos(\theta)}{a_R}), \]
\( \chi_{\rho} \) is a smooth cutoff function equal to one on \((-\frac{\beta}{2}, \frac{\beta}{2}) \times (-1, 1)\), and similarly \( \chi_{\rho} \) to one on \((-\frac{\beta}{2}, \frac{\beta}{2}) \times (-\ln(a_m), 0)\). Since the function defined by (4.5) is \( C^\infty \) for \( s \in [-1, 1] \), and correspondingly \( \rho \in [-\ln(a_m), 0] \), it is possible to choose \( \chi_{\rho} \) such that the two norms above are equivalent. It is clear that the results in §A is also valid with the norms above. By abuse of notation, we will denote all constants in the remainder of this section by \( C \).

The error introduced in the first interpolation step can now be expressed as
\[ \|I_{\Delta_1} h^\nu - h^\nu\|_{H^\gamma_{\rho}} \leq C|\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^\gamma_{\rho}}, \]
by using (A.8) of Theorem A.1. In the next step we make use of the Cauchy-Schwarz inequality in combination with the norm equivalence between \( \| \cdot \|_{H^\gamma_{\rho}} \) and \( \| \cdot \|_{H^\gamma} \),
\[ \|h^\nu - h^\nu\|_{H^\gamma_{\rho}} = \|I_{\Delta_1} I_{\Delta_2} h^\nu - h^\nu\|_{H^\gamma_{\rho}} \leq ||I_{\Delta_2} I_{\Delta_1} h^\nu - I_{\Delta_1} h^\nu\|_{H^\gamma_{\rho}} + ||I_{\Delta_1} h^\nu - h^\nu\|_{H^\gamma_{\rho}} \]
\[ \leq C|\Delta_2|^{\alpha_2-\gamma}|I_{\Delta_1} h^\nu|_{H^{\gamma_{\rho}}} + C|\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^\gamma_{\rho}}. \]
Since \( I_{\Delta_1} \) is bounded, it follows that
\[ ||h^\nu - h^\nu\|_{H^\gamma_{\rho}} \leq C(|\Delta_2|^{\alpha_2-\gamma}|h^\nu|_{H^{\gamma_{\rho}}} + |\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^\gamma_{\rho}}). \]
Hence, for the partial back-projection
\[ ||\mathcal{R}_1^\nu (h^\nu - h^\nu)\|_{H^\gamma_{\rho}} \leq \beta C(|\Delta_2|^{\alpha_2-\gamma}|h^\nu|_{H^{\gamma_{\rho}}} + |\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^\gamma_{\rho}}), \]
since both \( h^\nu \) and \( h^\nu \) are zero outside an \( \theta \)-interval of length \( \beta \). The total error from one partial back-projection is then bounded by
\[ ||\mathcal{R}_1^\nu (h^\nu - h^\nu)\|_{H^\gamma_{\rho}} \leq ||I_{\Delta_1} \mathcal{R}_1^\nu (h^\nu - h^\nu)\|_{H^\gamma_{\rho}} + ||\mathcal{R}_1^\nu (h^\nu - h^\nu)\|_{H^\gamma_{\rho}} \]
\[ \leq C|\Delta_1|^{\alpha_1-\gamma}|\mathcal{R}_1^\nu I_{\Delta_1} h^\nu|_{H^{\gamma_{\rho}}} + \beta C(|\Delta_2|^{\alpha_2-\gamma}|h^\nu|_{H^{\gamma_{\rho}}} + |\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^\gamma_{\rho}}) \]
\[ \leq C(|\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^{\gamma_{\rho}}} + |\Delta_2|^{\alpha_2-\gamma}|h^\nu|_{H^{\gamma_{\rho}}} + |\Delta_1|^{\alpha_1-\gamma}|h^\nu|_{H^\gamma_{\rho}}), \]
and hence also the total back-projection. We conclude the analysis above in the following theorem
\[ \text{Theorem 4.1. The reconstruction error made in the back-projection step of algorithm 4.1 satisfies} \]
\[ ||\mathcal{R}_1^\nu - \mathcal{R}_1^\nu\|_{H^\gamma_{\rho}} \leq C(|\Delta_1|^{\alpha_1-\gamma}|h|_{H^{\gamma_{\rho}}} + |\Delta_2|^{\alpha_2-\gamma}|h|_{H^{\gamma_{\rho}}} + |\Delta_1|^{\alpha_1-\gamma}|h|_{H^\gamma_{\rho}}). \]
We end up by a quantitative remark on the estimates above. The frequency content of (direct) quasi-interpolators is, generally speaking, one at a neighborhood of zero and dies out for higher frequencies, cf. proposition A.3. In principle they mimic sinc-interpolation, where the frequency content of the kernel is a box function centered at the origin. Thus, the error introduced by replacing the operator $\mathcal{R}$ with $I_{\Delta}, \mathcal{R}^\# I_{\Delta}$ is mainly due to what happens with Radon data far away from the origin. Now, consider the back-projection kernel of figure 3.1. Apart from along the line $\mu = 0$, most of its energy lies in a neighborhood of the origin. As pointed out above, if data (limited to $s \in [\ln(a_m), 1]$) is smooth with respect to polar coordinates, it is smooth also in the log-polar representation. The fact that data is band-limited with respect to $s$ in combination with the fast decay in the $\omega$-direction should therefore keep the errors relatively small in practise.

4.3. Time complexity and implementation. Let us analyze algorithm 4.1 slightly more in detail. The first filtering step is the same as in other filtered back-projection algorithms, and can be implemented by $p$ FFT operations of length $4q$ in time $\mathcal{O}(pq \log(q))$. At the first interpolation step, each interpolation $(I_{\Delta}, h^\nu)(\theta_j, \rho_i)$ consists of a weighted sum of $h(\theta, s_1)$ at points with $s_1$ within a kernel length distance to $\rho_i$. Hence this step is $\mathcal{O}(dpq') = \mathcal{O}(2d_3pqsq)$, with $d_1$ the kernel length of $I_{\Delta}$. Similarly, each interpolation from log-polar coordinates to cartesian ones is made by a weighted sum, in both $\theta$ and $\rho$-directions, giving a time complexity of $\mathcal{O}(4d_3q^2)$, where $d_3$ is the kernel area of $I_{\Delta_1}$, if reconstruction is made on a $(2q + 1) \times (2q + 1)$-grid. Assuming both $d_1$ and $d_3$ to be relatively small, and using the optimal relation between $p$ and $q$ in parallel beam geometry, $p = \pi q$, cf. [14], we arrive at a time complexity of $\mathcal{O}(q^2)$ for both interpolation steps.

Note that for both the interpolation weights above as well for the computation of the kernel $Z$ defined in (4.3), only geometry matters, i.e. these can be pre-computed to save time. What remains are then the two-dimensional FFT steps. Taken into account the needed zero-padding, $2m$ two-dimensional FFT operations of size $(2p \times q')$ are required. This is performed in time $\mathcal{O}(pq \log(pq)) = \mathcal{O}(q^3 \log(q))$, i.e. at the same complexity as other fast reconstruction methods. However, in comparison with e.g. Fourier slice-reconstruction on a $(2q + 1) \times (2q + 1)$ grid, the preceding constant of the leading term is worse. In principle this is due to over-sampling factors of $\kappa$ and $2$ in the $s$ and $\theta$ directions, respectively, and that both transformation and inversion are used. Together, this cause a worsening of a factor of about ten. However, that is in comparison with the most simplest Fourier slice-reconstruction, with its well known severe drawbacks in quality, and without effort of speeding up our proposed method (it is possible to use the zero-padded structure to decrease needed calculations). It should be added that that more sophisticated slice-reconstruction schemes also require over-sampling for adequate results.

Throughout §4.1 we worked with direct quasi-interpolators. The structure of algorithm 4.1 easily incorporates usage of the pre-filtering required by e.g. spline interpolators discussed in §A. Required pre-filtering for $I_{\Delta_1}$ can thus be included in $w_3$ of (4.1), and pre-filtering needed for $I_{\Delta_1}$ and $I_{\Delta_2}$ can be included in $Z$ defined by (4.3). This allows usage of spline interpolators without increasing computational cost, and thus allows higher interpolation order than the ones achieved by direct interpolators of the same kernel length.

4.4. Simulations. Finally, let us look at some numerical results. In these simulations we have used cubic spline interpolators for $I_{\Delta_1}, I_{\Delta_2}$ and $I_{\Delta_3}$, $m = 4$ partial reconstructions, $m_3 = 3$, and $\kappa = 2$. The number of parallels and directions are
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\[ q = 512 \text{ and } p = 768 \text{ respectively. Furthermore, to limit the regularization impact from the choice of filter in the filtering step, the Ram-Lak-filter suggested by Ramachandran and Lakshminarayanan in [17] is used.} \]

In e.g. [10], analytic expression for the Radon transform of functions being characteristic function of ellipses are derived. We use such functions as reconstructions objects, since they allow us to sample the Radon data exactly.

In the section we study two types of reconstruction. First we use characteristic function of small circles as approximations of delta functions, and then the Shepp-Logan phantom, described in [10], an artificial slice of a head built up by characteristic functions of ellipses.

Below are shown impulse responses at four different points for log-polar and classically computed back-projection, respectively. The reconstructions show quite high resemblance. Reconstruction on a cartesian grid makes the impulse response vary slightly depending on the placement of the impulse. It is worth noting that the responses usually are wider and smoother, when the ordinary filters of the filtering step are used.

Next we turn our attention to the Shepp-Logan phantom, cf. figure 4.2. The true head phantom carries an intensity of 1.02 at its inner, a border of intensity 2.0, “eyes” at 1.0 and additional ellipses at 1.03 (except at intersections). The two top images of figure 4.2 show reconstruction by log-polar (right) and traditional (left) back-projection. These seem almost identical, and it is hard to draw further conclusions. For a better perspective, we have chosen to plot the two reconstructions along a horizontal line through the “mouth” part of the head, at the center of figure 4.2. More

\[ \text{Impulse responses for log-polar (above) versus classical back-projection (below) at points (0.1233, 0.3816), (0.0998, 0.0998), (0.8004, −0.0020) and (−0.3229, −0.2368) respectively from left to right.} \]
precisely the values of the two reconstructions along row 410 has been plotted, with an offset of -0.001, to discriminate them. The two reconstructions display almost the same appearance, except minor disparities at the points of discontinuity. For further comparison, differences between the two reconstructions are shown in figure 4.2 below, with different intensity scales. The left one display differences in the reconstruction of the high intensity border around the skull. At right we can also see differences at the other discontinuity parts, but note the low intensity span. This image also clearly shows a slightly inhomogeneous assembling of lines outside the head, an affect of the different techniques used.

5. Conclusion. In this paper we have made use of the invariance properties of the Radon transform and its dual to construct a method of inversion based on a log-polar representation. An analysis of the continuous case has been presented as well as analytical expressions for the kernels. To deal with the non-uniformity of the discrete case, the concept of partial reconstructions was introduced. The non-uniformity also enforces the discrete reconstruction developed to rely on interpolation.
Fortunately, these can be invoked in the reconstruction procedure, for construction of a fast and accurate back-projection algorithm, based on use of the (uniform) two-dimensional FFT. The algorithm presented has a time complexity of $O(q^2 \log(q))$, the same as other fast reconstruction algorithms. From the tests presented in this paper, its accuracy seems to be practically the same as the one traditional back-projection algorithms have.

**Appendix. Some properties of convolution interpolators.** To obtain the error estimates needed above, some results on classical interpolation and sampling theory are needed. Since the focus of this article lies on practical tomography reconstruction, we’ll settle with qualitative results.

According to Whittaker-Shannon’s sampling theorem, for any $b$-band-limited function $f$, i.e. a function with a Fourier transform $\hat{f}(\xi)$ vanishing for $|\xi| > b$, it is possible to reconstruct $f$ given the uniformly sampled values of $f$ at $x_k = kT$, where $T = \pi/b$.

The reconstruction formula reads as

$$f(x) = \sum_{k \in \mathbb{Z}} f(x_k) \text{sinc}(x - x_k). \quad (A.1)$$

A drawback of this sinc-based interpolation is the slow decay of the sinc-function. The sinc-interpolation above is an example of convolution-based interpolation for uniformly sampled data:

$$I_T f(x) = \sum_{j \in \mathbb{Z}} c_j[f] \varphi\left(\frac{x}{T} - j\right); \quad (A.2)$$

a mixed convolution of a set of coefficients and an interpolation kernel $\varphi$. The coefficients $c_j[f]$ are generally obtained from

$$c_j[f] = \mu(f(\frac{x}{T} + j)), \quad (A.3)$$

where $\mu$ is a linear functional. For an overview of convolution-based interpolation, c.f. [13]. The operator $I_T$ is said to be an interpolator (sometimes referred to as cardinal interpolator) if $I_T f$ coincides with $f$ at the sample points.

In some cases it is of more interest if $I_T$ manages to reconstruct polynomials of a certain order correctly. To this end, $I_T$ is said to be a quasi-interpolator of order $p+1$ if it successfully interpolates all polynomials of degree $p$. The special simple case where $c_i[f] = f(x_i)$ is important, referred to as direct quasi-interpolators. A commonly used interpolator is the Key’s short cubic interpolator [11], with interpolation kernel

$$\varphi(x) = \begin{cases} 
(a + 2)|x|^3 - (a + 3)|x|^2 + 1, & 0 \leq |x| < 1; \\
(a|x|^3 - 5|x|^2 + 8|x| - 4), & 1 \leq |x| < 2; \\
0, & \text{otherwise},
\end{cases} \quad (A.4)$$

parameterized by $a$. Since $\varphi(x)$ is zero at a cardinal points it follows that Keys’s short cubic interpolator really is an interpolator.

The most widely used indirect interpolators are the spline interpolators, [18]. The B-spline kernel of order $m$ is defined by

$$\varphi_{B^m} = \left(\frac{1}{2} \chi(-\frac{1}{2}) + \frac{1}{2} \chi(-\frac{1}{2})\right) * \cdots * \left(\frac{1}{2} \chi(-\frac{1}{2}) + \frac{1}{2} \chi(-\frac{1}{2})\right) \quad (m + 1 \text{ factors}). \quad (A.5)$$

It is readily verified that $\varphi_{B^m}$ consists of piecewise polynomials of degree $m$, and that it is supported in $(-\frac{m+1}{2}, \frac{m+1}{2})$. 

Regarding the error of convolution-based interpolation, the Strang-Fix conditions plays an important role. The error estimates are carried out in a Sobolev framework, where $H^\gamma(\mathbb{R}^n)$ or $H^\gamma$ of real order $\gamma$ consists of the functions $f$ such that

$$
\|f\|_{H^\gamma}^2 = \sum_{|\alpha| \leq \gamma} \|D^\alpha f\|^2_{L^2(\mathbb{R}^n)} \text{ is finite,} \quad (A.6)
$$

where

$$
D^\alpha f = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}, \ \alpha = (\alpha_1, \ldots, \alpha_n).
$$

For the restriction of $H^\gamma_0$ to compactly supported functions, $H^\gamma_0$ is used as notation.

The following theorem can be found in [19] Theorem A.1. (Strang-Fix theorem)

Let $p$ be a positive integer and suppose $\varphi \in H^p_0$. Then the following conditions are equivalent:

1. $\hat{\varphi}(0) \neq 0$, but $\hat{\varphi}$ has zeros of order at least $p+1$ at the other points of $2\pi\mathbb{Z}^n$:

$$
D^\alpha \hat{\varphi}(2\pi j) = 0 \text{ if } 0 \neq j \in \mathbb{Z}^n, |\alpha| \leq p. \quad (A.7)
$$

2. for $|\alpha| \leq p$,

$$
\sum_{j \in \mathbb{Z}^n} j^\alpha \varphi(x - j)
$$

is a polynomial in $x_1, \ldots, x_n$ with leading term $\hat{\varphi}(0)t^\alpha$.

3. For each $f \in H^{p+1}$ there are coefficients $c_T^j$ such that as $|T| \to 0$,

$$
\|f - \sum_{j \in \mathbb{Z}^n} c_T^j \varphi(T^{-1} - j)\|_{H^s} \leq C_s |T|^{p+1-s} \|f\|_{H^{p+1}}, \ 0 \leq s \leq p, \quad (A.8)
$$

where $C_s$ is independent of $f$. Furthermore, these coefficients are bounded by

$$
\sum_{j \in \mathbb{Z}^n} |c_T^j|^2 \leq K \|f\|^2_{H^p}, \quad (A.9)
$$

where $K$ is independent of $f$.

Above, it is required that the kernel $\varphi$ has compact support, a requirement that can replaced by a decay condition, c.f. [7]. These conditions indicate that given some kernel $\varphi$ satisfying (A.7) of order $p+1$, it is possible to construct a (pseudo)-interpolator satisfying (A.8), i.e. approximating of order $p+1$. Under the right assumptions, this is true, but not a direct consequence of theorem A.1.

The characterization of the conditions needed on the functional $\mu$ in (A.3) is out of the scope of this paper, c.f. [6]. For our purposes it suffices be able to deal with direct quasi-interpolators. The following theorem is a simplified version of theorem 4.1 in [9]:

Theorem A.2. Let $I_T$ be a direct quasi-interpolator of order $p+1$, equipped with kernel $\varphi$ with sufficient decay, and let $f \in H^{p+1}$. Then

$$
\|f - I_T f\|_{H^p} \leq C|T|^{p+1} \|f\|_{H^{p+1}}. \quad (A.10)
$$
The following proposition for direct quasi-interpolators is easily shown by use of Poisson’s summation formula:

**Proposition A.3.** Let \( I_T f(x) = \sum_j f(jT) \varphi(x - jT). \) Then \( I_T \) is a quasi-interpolator of order \( p + 1 \) if and only if

\[
\begin{aligned}
\varphi(2\pi j) &= \delta[j], \\
D^\alpha \varphi(2\pi j) &= 0, & j \in \mathbb{Z}^n \text{ and } |\alpha| \leq p.
\end{aligned}
\]

(A.11)

where \( \delta[j] \) denotes the (discrete) Kronecker delta function.

Note that condition (A.7) differs slightly from (A.11), since (A.7) imposes no requirements on the derivatives of \( \varphi \) at the origin. It is straightforward to show that Key’s short cubic interpolator satisfies the Strang-Fix conditions up to quasi-interpolating order 3, if the free parameter is chosen as \( a = \frac{1}{2} \).

For \( \mathbb{R}^1 \), the simplest interpolation kernel which satisfies (A.7) is the B-spline kernel, since

\[ \varphi_{B^m}(\xi) = (\sin(\xi/2)/\xi)^m. \]

Thus, every \( \varphi \) which satisfies (A.7) is a multiple of \( \varphi_{B^m}, \varphi = \varphi_{B^m}E, \) where \( E \) is a suitable entire function, and hence every other interpolation kernel can be obtained as a convolution of a B-spline kernel.

However, it is not possible to construct a direct quasi-interpolator with \( \varphi_{B^m} \) as interpolation kernel for \( m > 2 \). This is due to the fact that the Taylor expansion of \( \varphi_{B^m} \) is

\[
\varphi_{B^m}(\xi) = 1 - \frac{(1 + m)\xi^2}{24} + O(\xi^4),
\]

(A.12)

which contradicts (A.11).

The natural question of how the mapping \( c_j[f] \) (i.e. the functional \( \mu \)) should be chosen for obtaining good (quasi)-interpolators now arises. The approach given here is based on the one outlined in [12]. From (A.2) follows that (in \( \mathbb{R} \)) for the corresponding \( I_T \) to be an interpolator,

\[
f(x_k) = \sum_{j \in \mathbb{Z}} c_j^{T} [f] \varphi(x - jT) = \sum_{j \in \mathbb{Z}} c_j^{T} [f] \varphi(k - j),
\]

(A.13)

must be satisfied. This (infinite) system of equations may be solved by means of the Z-transform, for which (A.13) becomes

\[
(\sum_{j \in \mathbb{Z}} \varphi(j)Z^j)C(Z) = F(Z).
\]

Looking for the impulse response, i.e. the response of \( f[k] = \delta[k] \), we obtain

\[
C_\delta(Z) = \frac{1}{\sum_{j \in \mathbb{Z}} \varphi(j)Z^j}.
\]

(A.14)

This can be used to construct a direct interpolator. Note that if \( I_T \) is a direct interpolator, then the function reconstructed from the sequence

\[ f[k] = \delta[k] \]
is the interpolation kernel. Using this yields
\[ \psi(x) = \sum_{j \in \mathbb{Z}} c_j^1 [\delta] \varphi(x - j) = \sum_{j \in \mathbb{Z}} c_j^1 [\delta] (\delta_j * \varphi)(x), \]
where \( \delta_k(x) = \delta(x - k) \) denotes the translated Dirac distribution. Hence,
\[ \hat{\psi}(\xi) = \left( \sum_{j \in \mathbb{Z}} c_j^1 [\delta] e^{-i\xi j} \right) \hat{\varphi}(\xi) = C_\delta(e^{-i\xi}) \hat{\varphi}(\xi). \quad (A.15) \]
For the sake of concreteness we consider the cubic spline interpolator, with interpolation kernel
\[ \varphi_{B^3}(x) = \begin{cases} 
(3|x|^3 - 6|x|^2 + 4)/6, & |x| \leq 1; \\
(2 - |x|)^3/6, & 1 \leq |x| \leq 2; \\
0, & \text{otherwise},
\end{cases} \]
explicitly calculated from (A.5). For this choice, the rightmost quantity of (A.13) becomes
\[ f(x_k) = \frac{1}{6} c_{k-1}^T[f] + \frac{4}{6} c_k^T[f] + \frac{1}{6} c_{k+1}^T[f]. \]
Equation (A.14) then becomes
\[ C_\delta(Z) = \frac{6}{Z + 4 + Z^{-1}}, \quad (A.16) \]
giving the impulse response
\[ c_j[\delta] = (-1)^k (2 - \sqrt{3})^j c_0[\delta], \quad c_0[\delta] = \frac{4}{3\sqrt{3}}. \quad (A.17) \]
Since \( 0 < 2 - \sqrt{3} < 1 \), the sequence \( c_k[\delta] \) oscillates and decays exponentially.
Combining (A.15) and (A.16) yields the direct interpolation kernel
\[ \hat{\psi}_{B^3}(\xi) = \frac{6}{4 + 2 \cos(\xi)} \left( \frac{\sin(\xi/2)}{\xi/2} \right)^4. \]
Hence, from the Taylor expansion
\[ \hat{\psi}_{B^3}(\xi) = 1 - \frac{1}{720} \xi^4 + O(\xi^6), \]
it follows from the requirement stated in (A.11) that the cubic spline interpolator constructed above is a quasi-interpolator of order 4. The advantage of such an interpolator, in contrast to e.g. the sinc interpolator of (A.1), is that due to the small support of the B-spline kernels, it is possible to perform the interpolation by using a small number of evaluations once the coefficients \( c_j \) has been calculated.
In this paper, two dimensional interpolators are also needed. Whenever so, tensor products of one dimensional interpolators, i.e.
\[ I_T f(x) = \sum_{j \in \mathbb{Z}^n} c_j^T[f] \varphi_1 \left( \frac{x_1}{T_1} - j_1 \right) \cdots \varphi_n \left( \frac{x_n}{T_n} - j_n \right), \]
are used. In such cases \( |T| = \max(T_1, T_2) \) in theorems A.1 and A.2.
REFERENCES

[18] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions. Part A and B.