The solids-flux theory – confirmation and extension by using partial differential equations

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Abstract
The solids-flux theory has been used for half a century as a tool for estimating concentration and fluxes in the design and operation of secondary settling tanks during stationary conditions. The flux theory means that the conservation of mass is used in one dimension together with the batch-settling flux function according to the Kynch assumption. The flux-theory results correspond to stationary solutions of a partial differential equation, a conservation law, with discontinuous coefficients modelling the continuous-sedimentation process in one dimension. The mathematical analysis of such an equation is intricate, partly since it cannot be interpreted in the classical sense. Recent results, however, make it possible to partly confirm and extend the previous flux-theory statements, partly draw new conclusions also on the dynamic behaviour and the possibilities and limitations for control. We use here a single example of an ideal settling tank and a given batch-settling flux in a whole series of calculations. The mathematical results are adapted towards the application and many of them are conveniently presented in terms of operating charts.

Key words: operating chart, continuous sedimentation, secondary clarifier, thickener, dynamic modelling, control

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>A</td>
<td>cross-sectional area [m²]</td>
</tr>
<tr>
<td>D</td>
<td>depth of thickening zone [m]</td>
</tr>
<tr>
<td>D</td>
<td>‘dangerous’ region in operating chart</td>
</tr>
<tr>
<td>E</td>
<td>excess flux in steady state [kg/(m²h)]</td>
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<tr>
<td>F</td>
<td>total flux function in Eq. (1) [kg/(m²h)]</td>
</tr>
<tr>
<td>H</td>
<td>height of clarification zone [m]</td>
</tr>
<tr>
<td>O</td>
<td>overloaded region in operating chart</td>
</tr>
<tr>
<td>Q</td>
<td>volumetric flow rate [m³/h]</td>
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<tr>
<td>Qₘₐₓ</td>
<td>maximum bound on control variable, cf. (10) [m³/h]</td>
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<tr>
<td>Qₘᵟᵢₙ</td>
<td>minimum bound on control variable, cf. (11) [m³/h]</td>
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<tr>
<td>S</td>
<td>‘safe’ region in operating chart</td>
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<tr>
<td>U</td>
<td>underloaded region in operating chart</td>
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<tr>
<td>X</td>
<td>total suspended solids concentration [kg/m³]</td>
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<tr>
<td>Xₐᵢᶠ</td>
<td>inflection point of flux functions [kg/m³]</td>
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<tr>
<td>Xₘₐₓ</td>
<td>maximum concentration [kg/m³]</td>
</tr>
<tr>
<td>Xₘᵢₙ</td>
<td>local maximum point of f [kg/m³]</td>
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<tr>
<td>Xₘ₃ᵢₙ</td>
<td>local minimum point of f [kg/m³]</td>
</tr>
<tr>
<td>Xₘ₃ₘ</td>
<td>concentration satisfying Xₘ₃ ≤ Xₗⁱᶠ</td>
</tr>
<tr>
<td>Xₘ₃ₘ</td>
<td>and f(Xₘ₃ₘ) = f(Xₘ₃) [kg/m³]</td>
</tr>
<tr>
<td>Xₘᵟᵢₙ</td>
<td>minimum bound on underflow concentration, cf. (9) [kg/m³]</td>
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<tr>
<td>f</td>
<td>flux function in thickening zone [kg/(m²h)]</td>
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<tr>
<td>f₀</td>
<td>batch-settling flux function [kg/(m²h)]</td>
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<tr>
<td>fₙₘᵢₚ</td>
<td>limiting flux function [kg/(m²h)]</td>
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<tr>
<td>fₙₘᵢₚ</td>
<td>flux in thickening zone in steady state [kg/(m²h)]</td>
</tr>
<tr>
<td>ℓ</td>
<td>line in operating chart</td>
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<tr>
<td>p</td>
<td>point in operating chart</td>
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<tr>
<td>q</td>
<td>volumetric flux [m³/h]</td>
</tr>
<tr>
<td>s</td>
<td>source term, flux fed to SST [kg/(m²h)]</td>
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<tr>
<td>t</td>
<td>time [h]</td>
</tr>
<tr>
<td>vₜₜₗₜ</td>
<td>settling velocity [m/h]</td>
</tr>
<tr>
<td>x</td>
<td>depth from feed level [m]</td>
</tr>
<tr>
<td>y</td>
<td>flux axis in operating chart [kg/(m³h)]</td>
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Greek letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>Λ</td>
<td>region in steady-state control chart</td>
</tr>
<tr>
<td>δ</td>
<td>Dirac delta distribution [1/m]</td>
</tr>
<tr>
<td>θ</td>
<td>= Xₐ/Xₖ thickening factor [-]</td>
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</table>

Subscripts

<table>
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<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>₀</td>
<td>initial value at t = 0</td>
</tr>
<tr>
<td>cl</td>
<td>clarification zone</td>
</tr>
<tr>
<td>e</td>
<td>effluent</td>
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1. Introduction

The complexity of the secondary settling tank (SST) within the activated sludge process is not only known to operators of the plants, but also to researchers. Interesting nonlinear phenomena are shown even if simplifying idealizations are made in models. The same experiences have been made in other fields, since clarifier-thickener units are also used in the mineral, chemical, pulp-and-paper and food industries.

Several simulation models have been suggested for clarifier-thickener units. This has been a natural development, particularly for the SST, because of its presence in the activated sludge process and the fact that there are reliable simulation models for the biological reactor.

However, even if reliable simulation programs are developed, they do not give general rules on how to control the process. Then one has to go back and investigate properties of the basic physical laws that govern the process. Such a law is the conservation of mass, which can be written exactly as an equation with integrals. Such a continuity equation can be reformulated and written equivalently as a partial differential equation (PDE), also called a conservation law, provided that it is interpreted in the so-called weak sense. This is a mathematical terminology, which means that the PDE is only a convenient symbol for an equation containing integrals. For example, a solution of a PDE may have discontinuities, despite the fact that such are not differentiable in the ordinary sense. This sometimes causes misunderstandings. Conclusions, or numerical algorithms, derived directly from the PDE may thus be incorrect.

Although different ways of tackling the problem have been presented during half a century by water researchers, chemical engineers, applied mathematicians, etc., a common platform has been the celebrated paper by Kynch (1952), which we may consider to be the origin of the solids-flight theory. Kynch’s constitutive assumption is that the local settling velocity is a function of the concentration only, $v_{\text{sett}}(X)$, and it is decreasing. Batch sedimentation in a column can in one dimension be described by the conservation law (interpreted in the weak sense)

$$\frac{\partial X}{\partial t} + \frac{\partial f_b(X)}{\partial x} = 0,$$

where $f_b(X) = Xv_{\text{sett}}(X)$ is the batch-settling flux function. It is assumed that this function has an inflection point and a typical graph is shown in Fig. 1. Kynch also showed how solutions could be constructed by the method of characteristics.

Fundamental results of graphical constructions by using the flux curve $f_b(X)$ for obtaining concentrations in steady-state operation were presented by Jernqvist (1965a,b,c). Unfortunately, Jernqvist’s results seem not to have reached other researchers. Similar developments, but not as extensive, were presented in the 1960s onwards with concepts such as the operating line, the limiting flux and the state point (pivot point, feed point), see e.g. Ekama et al. (1997); Diehl (2001) and references therein. Interpretations concerning clarification failure and control were made by Keinath et al. (1977); Laquidara and Keinath (1983); Keinath (1985) and a chart describing steady states was presented by Lev et al. (1986). Later references also showed the need for the flux theory for describing steady states of the process (e.g. Chancelier et al., 1997a; Ekama and Marais, 2004; Kaushik and Murthy, 2002; Lynggaard-Jensen and Lading, 2006; Narayanan et al., 2000; Wett, 2002; Wilén et al., 2004).

At the same time, one should be aware of the inherent limitations of a one-dimensional ideal model together with only one constitutive assumption (Kynch’s). The complexity of two- and three-dimensional models makes, however, a one-dimensional model still interesting for the design of full-scale SSTs. Ekama and Marais (2004) then recommend a safety factor (reduction factor) to account for the hydraulic non-idealities.

A first-order PDE model of the entire SST was presented independently by Chancelier et al. (1994) and Diehl (1995, 1996). It is a one-dimensional model together with Kynch’s constitutive assumption. Hence, solutions of the PDE are physically correct solutions within the flux theory. Consequently, the flux theory can be confirmed and extended in a natural way within the context of PDE theory. This was done by Chancelier et al. (1997a,b). It is the aim of the present paper to present further results in this direction. We emphasize that ‘a physically correct solution’ refers to the physical law, or equivalently, the PDE and the constitutive assumption. To what extent such a solution approximates the real concentrations of a SST is another issue not dealt with here.

The inlet and outlets of the SST cause differing fluxes and conditions, which imply that coefficients in the PDE are space discontinuous. These discontinuities imply a severe nonlinear feature in addition to the nonlinear sedimentation-consolidation process itself. This has implied that basic research in mathematics has had to be carried out in order to obtain a correct description of the process and to develop reliable numerical methods, see the special issue on conservation laws with discontinuous flux with the editors Bürger and Karlsen (2008).

In a series of papers (Diehl, 2001, 2005, 2006, 2008a) thorough mathematical analyses of the first-order PDE model have been performed. It is the purpose of the present paper to extract, adapt and discuss some of the information

| $t$ | feed (inlet) |
| $h$ | thickening zone |
| $n$ | underflow |

Overscripts

- related to inflection point
- related to critically loaded SST
- related to optimal operation
from this series of papers that confirms and extends the solids-flux theory. The reader is referred to that series for all details and proofs.

2. The first-order model of an ideal SST

Basically, the first-order one-dimensional model of an ideal SST is the conservation of mass together with Kynch’s assumption. Let the x-axis be directed downwards with the feed level located at $x = 0$. Denote the depth of the thickening zone by $D$, the height of the clarification zone by $H$ and the constant cross-sectional area by $A$. Then the conservation of mass yields the following conservation law, which should be interpreted in the weak sense, (Diehl, 1996)

$$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x}(F(X,x,t)) = s(t)\delta(x), \quad (1)$$

where the total flux function is

$$F(X,x,t) := \begin{cases} -q_e(t)X, & x < -H \\ f_b(X) - q_e(t)X, & -H < x < 0 \\ f_b(X) + q_a(t)X, & 0 < x < D \\ q_a(t)X, & x > D, \end{cases}$$

with the volumetric fluxes $q_e = Q_e/A$ etc., and the source function is

$$s(t) := q_e(t)X_t(t) = (q_a(t) + q_e(t))X_t(t). \quad (2)$$

For the flux function in the thickening zone we define

$$f(X,Q_a(t)) := f_b(X) + q_a(t)X,$$

but it is often convenient to suppress the dependence on the volumetric flux, i.e. we write $f(X)$.

Note that Eq. (1) is comprehensive. Inside the thickening zone, for example, it reduces to $\frac{\partial^2 X}{\partial x^2} + \frac{\partial f(X)}{\partial x} = 0$.

It is assumed that the given input variables are $X_t$ and $Q_t$, or rather, $X_t$ and $s$. The solution $X(x,t)$ of (1) is to be determined by the model, and it contains the outlet concentrations $X_H$ and $X_D$. It is assumed that the volumetric underflow rate $Q_a$ can be used as a control variable. Hence, the effluent volumetric flux rate is given by $Q_e = Q_t - Q_u$.

To adjust the model to real data, we may also have parameters in the constitutive relation, i.e. the batch-settling flux. However, this is not the topic here. We merely assume that a sustainable batch-settling flux function $f_b$ has been determined for the sludge under consideration.

To facilitate further reading with more examples and simulations in the series (Diehl, 2001, 2005, 2006, 2008a), the same numerical data and batch-settling flux function are used here (see Fig. 1). Of course one can use the Vesilind formula (Vesilind, 1968) or some other instead. The results will be qualitatively the same regardless of what formula for the settling velocity is used (as long as the graph of the flux function is qualitatively as in Fig. 1). For numerical simulations it is convenient to have a maximum concentration and we have set $X_{\text{max}} = 10$ kg/m$^3$. As for the dimensions of the SST, we assume that the depth of the thickening zone is $D = 4$ m, the height of the clarification zone is $H = 1$ m and the cross-sectional area is $A = \pi(30)$ m$^2$ $\approx 2827$ m$^2$.

The numerical simulations are performed with the method in Diehl and Jeppsson (1998).

In Fig. 1(a), the flux function in the thickening zone is drawn for two values of $Q_u$. We let $Q_u$ denote the specific value of $Q_a$ above which the flux function is strictly increasing, and we set $Q_u = Q_a/A$.

3. The flux theory and operating charts

The classical solids-flux theory is described and commented in several papers (e.g. Chancelier et al., 1997a; Ekama et al., 1997; Jernqvist, 1965a,b,c; Keinath, 1985; Ozinsky et al., 1994). Examples of graphical constructions are shown in Fig. 1(b)–(c). The intersection of the ‘operating line’ $y = s - q_a u$ and the ‘effluent line’ $y = q_e u$ defines the ‘state point’. This occurs at the concentration $X_t$ because of (2). If, for intermediate feed concentrations, the operating line cuts the batch flux curve two or three times, then it is known that the SST will become overloaded. This is called thickening failure and occurs thus for $s - q_a X_M > f_b(X_M) \iff s > f(X_M)$, where $f(X_M)$ is called the limiting flux. Other steady-state situations are also described in the literature, however not all (corresponding to an arbitrary load, i.e. an arbitrary location of the state point).

If the operating line moves above the whole batch flux curve, or intersects it only once at a high concentration, the situation is called clarification failure (Laquidara and Keinath, 1983). Thickening and clarification failure are sometimes referred to as transient situations, since the movements of discontinuities are interesting. To describe such dynamic situations, the platform of PDE theory is appropriate and yields more information as we shall see.

The concept of state point is slightly misleading, because it neither describes the state of the SST at a particular instant in time, nor (necessarily) corresponds to a unique asymptotic state after a long time with constant feed variables, since this depends on the initial concentration distribution in the SST. Therefore, we suggest the alternative notion feed point for the pair of feed variables $(X_t,s)$.

In the classical concentration-flux diagram in Fig. 1(c), the concentrations in the thickening zone and the underflow concentration can be obtained along the operating line, since the following holds in steady state:

$$s = q_e X_t = f(X_M) - f(X_M) = q_a X_u \quad \text{(critical loading)}.$$

Since these three fluxes are equal, it is more natural to have the operating line horizontal in a concentration-flux diagram. Adding the volumetric flux $q_e X$ to the graphs of $f_b(X)$ and the straight lines in Fig. 1(c), we get Fig. 1(d), which we call an operating chart. Then the previous operating line becomes horizontal at the flux value $s$ that is present at every depth in the thickening zone for an underloaded or critically loaded SST. The previous effluent line
becomes \( y = q_f X \), which we call the feed line. In this operating chart, the values of the feed point \((X_f, s)\) can be read off on the axes in the usual sense. Note that the feed point always lies above the underflow line \(y = q_uX\), since \(s = q_fX - (q_u + q_e)X_f > q_uX\) (assuming \(q_e > 0\)).

4. An operating chart for steady states

For any location of the feed point \((X_f, s)\) in the operating chart, there is one or several corresponding steady-state solutions. Each steady-state solution is piecewise constant and non-decreasing with depth in this first-order model. The qualitatively different ones are conveniently described by the steady-state chart, see Fig. 2 (left). The regions are divided by the aid of the graph of the flux function \(f\), straight lines and the limiting flux function \(f\) (Chancelier et al., 1997a):

\[
f_{\text{lim}}(X) := \min_{X \leq \alpha \leq X_{\text{max}}} f(\alpha) = \begin{cases} f(X), & X \in [0, X_m] \cup [X_M, X_{\text{max}}], \\ f(X_M), & X \in (X_m, X_M). \end{cases}
\]

Thus, the classical limiting flux value \(f(X_m)\) is extended to a function. It is only for feed concentrations between \(X_m\) and \(X_M\) the limiting flux function is equal to \(f(X_m)\).

Each region in Fig. 2 (left) corresponds to a specific type of steady state. An accompanying table of all steady states can be found in Diehl (2001). It turns out that for any location of the feed point \((X_f, s)\) in the chart, the following formula holds in steady state for the flux in the thickening zone and the underflow concentration:

\[
f_{\text{thick}}(X_f, s) = \min\left(s, f_{\text{lim}}(X_f)\right),
\]

\[
X_u(X_f, s) = \frac{f_{\text{thick}}(X_f, s)}{q_u}.
\]

Define the excess flux as

\[
\mathcal{E}(X_f, s) := s - f_{\text{lim}}(X_f).
\]

If the feed point lies above the graph of the limiting flux, then \(\mathcal{E}(X_f, s) > 0\) is the excess flux that the thickening zone cannot handle. This flux of sludge is then transported upwards through the clarification zone and the SST is overloaded and the effluent concentration is positive. The following formula holds for the effluent concentration in steady state as a function of any feed point:

\[
X_e(X_f, s) = \frac{\max\{0, \mathcal{E}(X_f, s)\}}{q_e} \quad (q_e > 0).
\]

If the feed point \((X_f, s)\) lies below the limiting flux curve, then \(\mathcal{E}(X_f, s) < 0\) holds. The effluent concentration is then zero and the SST is underloaded since it can handle more feed flux with still zero effluent concentration. The intermediate case is when the feed point is located on the limiting flux curve. Then \(\mathcal{E} = 0\) and the SST is critically loaded.

For example, if the feed point lies in the underloaded region \(\mathcal{U}_1\), then the concentration in the effluent and the clarification zone is zero \((X_e = 0, X_{cl} = 0)\), there is a constant concentration \(X_u\) in the thickening zone and the underflow concentration is \(X_u = s/q_u\) according to Fig. 3 (left). All the incoming flux is conveyed through the thickening zone.

In Fig. 3 (right), an overloaded situation is shown. All concentrations appearing in the solution can be obtained by drawing the auxiliary line \(y = q_fX - \mathcal{E}\), which is long-dashed in Fig. 3 (right) and parallel to the feed line.

As the feed point lies in the region \(\mathcal{O}_1\) or in \(\mathcal{O}_3\) (in Fig. 2 (left)), it turns out that the steady-state solution consists of a constant, equal to the feed concentration, in both the clarification and thickening zones.

The stationary solution is in fact uniquely determined by the feed point except when this lies on the limiting flux curve or on the lines \(\ell_3\) or \(\ell_5\). Then the location of a discontinuity in the thickening or the clarification zone is not uniquely determined. This discontinuity corresponds to the sludge blanket level (SBL).

The most wanted steady state is when the SBL is located in the thickening zone, with the concentration \(X_m\) above and \(X_M\) below, and there is zero concentration in the clarification zone. Then the SST is in optimal operation in steady state. This can only occur as the feed point lies on the line segment \(p \cup \ell_2 \cup \ell_3\) in Fig. 2. Unfortunately, this is not sufficient for optimal operation. If the feed point lies on this line segment, it may also be the case that the SBL is located at the feed level or at the bottom of the tank. Then there is really no SBL. This is because the actual asymptotic state depends on the initial concentration distribution in the SST. It is worth noting that in optimal operation in steady state, the feed concentration is diluted just below the feed inlet, \(X_m \leq X_f\). Furthermore, as \((X_f, s) \in \ell_3\), then \(X_M < X_f < X_u\) holds, i.e. the feed concentration is also higher than the one below the SBL.

5. Control of steady states

Many of the fluxes defined above depend on the control variable \(Q_u\), and we may write out this dependence when necessary; e.g. \(f(X, Q_u), f_{\text{lim}}(X_f, Q_u)\) and \(\mathcal{E}(X_f, s, Q_u)\). This means that all regions of the charts in Fig. 2 also depend on \(Q_u\). The local minimum point of \(f, X_M(Q_u)\) (see Fig. 1(a)), decreases with \(Q_u\), and the associated smaller concentration with the same flux value, \(X_m(Q_u)\), increases. In Fig. 4 (left), several limiting flux curves are drawn for different values of \(Q_u\). This figure shows the dependence of \(X_M(Q_u)\) and \(X_m(Q_u)\) as \(Q_u\) varies. Two curves are formed, which are drawn in Fig. 4 (right). As \(Q_u\) reaches up to the specific value \(\bar{Q}_u\), these two curves meet at the point of inflection of the flux function. Consequently, the set \(\mathcal{A}_2\) is precisely the region that the line segment \(\ell_2(\bar{Q}_u)\) (in Fig. 2) passes as \(Q_u\) varies between 0 and \(\bar{Q}_u\). The other regions in Fig. 4 (right) are defined analogously \((\mathcal{A}_1\) is the region that the line segment \(\ell_1(\bar{Q}_u)\) passes, etc.).

Note that the regions in Fig. 4 (right) are fixed and do not depend on \(Q_u\). We call this the control chart for steady states. The reasons are the following two facts on the pos-
in the clarification zone, the thickening zone. (c) Critical conditions occur as the operating line is tangent to the batch flux curve at the inflection point, local maximum and minimum points. The graphs (b)–(c) show classical graphical constructions using $f_b$:

- (b) Underloaded conditions where $X_{th}$ is the constant concentration in the thickening zone.
- (c) Critical conditions occur as the operating line is tangent to the batch flux curve at $X_M$. There is a zero concentration in the clarification zone, $X_m = 0$, hence $X_v = 0$, and there is a (possible) sludge blanket in the thickening zone with the concentration $X_m$ above and $X_M$ below it.
- (d) Preferred operating chart instead of the one in (c).

**Fig. 1.** The batch-settling flux used for the graphs and simulations is $f_b(X) = 10X \times (1 - 0.64X/X_{max})^{0.55 - 0.36^{0.55}} \times [\text{kg/m}^2\text{h}]$, where $X_{max} = 10 \text{kg/m}^3$. The inflection point is $X_{infl}$.

- (a) The flux function in the thickening zone for two values of the control variable and some characteristic concentrations, such as the inflection point, local maximum and minimum points.
- The graphs (b)–(c) show classical graphical constructions using $f_b$: (b) Underloaded conditions where $X_{th}$ is the constant concentration in the thickening zone. (c) Critical conditions occur as the operating line is tangent to the batch flux curve at $X_M$. There is a zero concentration in the clarification zone, $X_m = 0$, hence $X_v = 0$, and there is a (possible) sludge blanket in the thickening zone with the concentration $X_m$ above and $X_M$ below it. (d) Preferred operating chart instead of the one in (c).

**Fig. 2.** Left: The steady-state chart. The thick graph is the limiting flux curve. If the feed point lies on this curve, the SST is critically loaded in steady state, which means that it works at its maximum capacity. If the feed point lies below this graph, in a region denoted by $U$, the SST is underloaded in steady state. Above the limiting flux curve, in a region denoted by $O$, the SST is overloaded in steady state with a non-zero effluent concentration. Note that the regions in this chart all depend on the control parameter $Q_u$ since $f$ does. Right: Operating chart for step responses from optimal operation. This means a subdivision of the steady-state chart, e.g. $O_2 = O_{2a} \cup O_{2b} \cup O_{2c}$. Each region corresponds to a specific type of step response (transient solution during $t > 0$) as the feed point jumps at $t = 0$ from $(X_{f0}, s_0) \in p \cup \ell_2 \cup \ell_3$ to that region.
sibility of controlling steady-state solutions:

− Given any location of the feed point \((X_f, s)\) in Fig. 4 (right), there exists a unique value on the control variable \(Q_u(X_f, s)\) such that \(\mathcal{E}(X_f, s, Q_u(X_f, s)) = 0\), i.e. the feed point lies on the limiting flux curve and the SST is critically loaded in steady state.

− Given a feed point \((X_f, s)\) ∈ \(\Lambda_2 \cup \Lambda_3^c\), there exists a unique value \(Q_u(X_f, s)\) such that \((X_f, s) \in \mathcal{E}_2(Q_u) \cup \mathcal{E}_3(Q_u)\), which is necessary for the SST to be in optimal operation in steady state.

The value \(Q_u(X_f, s)\) implies that the SST may be in optimal operation asymptotically if the feed and control variables are kept constant. That is, there may be a SBL within the thickening zone with \(X_m\) above and \(X_M\) below it. The two extreme solutions with either \(X_m\) or \(X_M\) in the entire thickening zone are also possible. Thus, we may control the qualitative type of steady state with the aid of the control chart in Fig. 4 (right). The location of the SBL in an asymptotic state depends on the initial concentration distribution and to control this location the dynamic behaviour needs to be incorporated.

There is a slight difference between the two specific values of the control variable, which may not be of a great importance for the application. If the plant has a sufficiently large cross-sectional area, then \(s = X_f Q_f/A\) is small and it is most likely that the feed point lies in \(\Lambda_2\). Then \(Q_u(X_f, s) = Q_u(X_f, s)\) holds. If \((X_f, s) \in \Lambda_3\), then \(Q_u(X_f, s) < Q_u(X_f, s)\) holds. The latter inequality means that the SST can be in optimal operation with the value of the control parameter equal to \(Q_u\), and still be slightly underloaded. However, as can be concluded from the control chart, this occurs only if the feed concentration is high (at least greater than \(X_{infl}\)) and/or the feed flux \(s\) is sufficiently high.

Another remark is that for a general batch-settling flux there is no explicit formula for neither \(Q_u\) nor \(Q_u\). For example, assuming that the feed point is in \(\Lambda_2\), then \(Q_u = Q_u\) is defined implicitly (as a function of \((X_f, s)\)) by

\[
\mathcal{E}(X_f, s, Q_u) = 0 \iff s = f_{\text{lim}}(X_f, Q_u) \iff s = f(X_M(Q_u), Q_u),
\]

where the local minimum point \(X_M(Q_u)\) is the largest solution of the equation

\[
\frac{\partial f(X, Q_u)}{\partial X} = 0.
\]  

Using, for example, the Vesilind formula for the settling velocity, it is impossible to eliminate \(X_M\) and solve for \(Q_u\) explicitly as a function of the feed point from the two equations (5) and (6). Hence, \(Q_u\) has to be obtained numerically generally. It is rather easy, however, to obtain graphical overviews. Since \(Q_u(X_f, s)\) is the solution of \(s = f_{\text{lim}}(X_f, Q_u)\), the contours of \(Q_u(X_f, s)\) are precisely the graphs shown in Fig. 4 (left).

Assume that the SST can be controlled such that it is critically loaded in steady state, i.e. the control variable has the value \(Q_u(X_f, s)\), and the SST works at its maximum capacity. This is the lowest possible value of \(Q_u\) for which there is no overflow. The corresponding maximum underflow concentration in steady state is then

\[
\tilde{X}_u(X_f, s) := \frac{sA}{Q_u(X_f, s)}.
\]

This is in accordance with the general formulae (3) and (4), but is directly obtained as a mass balance, since all flux is conveyed through the thickening zone. The contours of this function are drawn in Fig. 5(a).

The maximum effluent volumetric flux, or surface overflow rate, for a critically loaded SST in steady state is

\[
\tilde{q}_u(X_f, s) := \frac{s}{X_f} - \frac{Q_u(X_f, s)}{A}.
\]

Contours of this function are shown in Figure 5(b).

Another interesting concept of the capacity of the SST is the thickening factor \(\tilde{\theta} := X_u/X_f\). The maximum thickening factor for a critically loaded SST in steady state is thus

\[
\tilde{\theta}(X_f, s) := \frac{X_u(X_f, s)}{X_f} = \frac{As}{Q_u(X_f, s) X_f}.
\]

This is shown in an operating chart in Fig. 5(c).

6. Step responses

As a first approach to analyze the dynamic behaviour of a system, step responses can be investigated. A linear time-invariant system with two input and two output signals is uniquely determined by its response to only two (linearly independent) step inputs. For a nonlinear system, a step response only yields information on the system’s behaviour for the specific input data. Several step responses may, however, give a good understanding of the system and yield information on different regions of the two-dimensional input-data space where the system shows qualitatively different behaviours. The process of continuous sedimentation can be seen as a nonlinear system with the feed variables \((X_f, s)\) as the inputs and the outlet concentrations \(X_e\) and \(X_u\) as the outputs. Since the process is governed by a PDE, there is also an infinite number of internal state variables. Exact dynamic solutions of the PDE (1) can be obtained with the method of characteristics together with the procedure at the inlet and outlets by the author (Diehl, 1995, 1996, 2000).

In this section we assume that the SST is initially in optimal operation in steady state, i.e. the feed point satisfies initially \((X_0, s_0) \in p(Q_0) \cup \mathcal{E}_2(Q_0) \cup \mathcal{E}_3(Q_0)\). In the numerical calculations we assume that the SBL is located in the middle of the thickening zone initially, i.e., at \(x = 2\) m. As the feed point makes a jump to another region in the operating chart, a transient solution occurs, a step response, and after a finite or infinite time a new steady state arises. The type of this new state is given by the steady-state chart in Fig. 2 (left). However, there are qualitatively different transients also within some of the regions of that chart. The
operating chart for step responses from optimal operation is shown in Fig. 2 (right).

As two examples, we show in Figs 6 and 7 numerical simulations of step responses. In both cases there is a rising discontinuity in the thickening zone from the initial stationary SBL, however, the behaviour in the clarification zone is qualitatively different. It is also clear that the total mass increases in both cases. This is true for step responses as the feed point jumps to any of the overloaded regions, except for \( O_1 \) and if \( s > s_0 = f(X_M) \). In this latter case the feed concentration has made a jump down at the same time as \( Q_f \) has made a step up (since \( s = q_f X_f \) has jumped up and \( X_f \) down). Then the mass increases initially during a small time period and then decreases to a lower value than the initial one. After a finite time, the concentration is equal to \( X_f \) in the whole SST. For solutions/simulations of all step responses we refer to Diehl (2005).

We have earlier defined optimal operation in steady state. It is convenient to generalize this to include also dynamic solutions, such as in Figs 6 and 7. We first define more precisely the SBL as the smallest depth at which the concentration passes \( X_{\text{min}} \). The SST is said to be in optimal operation at time \( t \) if the following holds:

1. \( Q_u(t) < \dot{Q}_u \), the concentration is zero in the clarification zone, and
2. there is a SBL in the thickening zone.

For example, in Fig. 6, optimal operation is left after 4.8 h, and in Fig. 7 optimal operation is left immediately.

Now we can review some of the comprehensive conclusions in Diehl (2005) regarding step responses. The main difference between the two step responses in Figs 6 and 7 demonstrate a common property that divides all step responses into only two categories. The operating chart for step responses in Fig. 2(b) can be divided into the ‘safe’ region \( S := Q_1 U_1 \cup Q_1 U_2 \cup U_1 U_2 \cup U_1 U_3 \) and the ‘dangerous’ region \( D := O_1 U_1 \cup U_2 \cup O_2 \cup O_3 \cup U_1 \cup U_2 \cup U_3 \cup U_1 U_2, \) see Fig. 8(a). If the feed point makes a step to

- \((X_f, s) \in S\), then the SST stays in optimal operation for a while and there is time for control actions.
- \((X_f, s) \in D\), then optimal operation is left directly – clarification failure occurs.

These facts illustrate the impact of the local maximum \( X_M \) (indicated in e.g. Fig. 8(a)) for dynamic solutions by analogy with the impact the local minimum \( X_M \) has for stationary solutions.

Besides the qualitative behaviours of step responses represented by Fig. 2(b) and Fig. 8(a), quantitative information can be obtained. As the feed point jumps down to \( U_1 \), the SBL will decline and reach the bottom after a finite time. This time depends only on the new applied flux \( s \), see Fig. 8(b).

As we have seen, the step responses that result in overflow can be divided into two major categories depending on whether the feed point stays within \( S \) or not, corresponding to thickening and clarification failure, respectively, and the simulations in Figs 6 and 7 are two such examples. Quantitative information on how long it takes until optimal operation is left, overflow occurs etc. are presented in terms of formulae and charts in Diehl (2005). Two such charts related to thickening and clarification failure are shown in Figs 8(c)–(d).

7. Optimal control of step responses

As the feed variables vary with time, a natural control objective is to maintain optimal operation as long as possible. If the SST is highly loaded, it is necessary to increase \( Q_u \) to avoid an overflow. However, such a control action implies that the underflow concentration decreases. An additional requirement in a control objective may be that the underflow concentration should not lie below a predefined minimum level. Hence, maintaining optimal operation and a sufficiently high underflow concentration simultaneously may not be possible after a certain time point if the SST is highly loaded. Another rule has then to be included into the control objective. One has to decide which is least important if all the requirements cannot be satisfied. Different such control objectives have been formulated in Diehl (2006) and optimal control strategies presented on how to fulfill different objectives for step inputs. By specifying a control strategy we mean that \( Q_u(t) \) is defined as a function of the varying \((X_f(t), s(t))\) and the present location of the SST. We shall here only give an example.

Assume that the SST initially is in optimal operation in steady state with a SBL close to the bottom. At \( t = 0 \) there is a step increase such that the feed point moves up into the dangerous region \( D \), i.e., a high load is applied. If no control action is performed, clarification failure occurs as in Fig. 7. The first remedy is to increase \( Q_u \) directly from its initial value \( Q_{u0} = 3500 \) m³/h such that the safe region \( S \) moves upwards and includes the new feed point. The lowest possible \( Q_u \) means that the feed point lies on the boundary between the regions \( S \) and \( D \). The solution is shown in Fig. 9. After a small initial decline the SBL rises. At \( t = 16 \) h the SST has almost reached the feed level and it is necessary to increase the control variable further to the value \( Q_u \approx 4298 \) m³/h. Then the system is critically loaded in its steady state and Fig. 9 shows that optimal operation is maintained with the SBL just under the feed level, however, the underflow concentration is reduced further.

In the example of a highly loaded SST in Fig. 9, there is no possibility of controlling the underflow concentration so that it lies above a prescribed level \( X_{\text{min}}^u \). This holds of course only if our control objective states that maintaining optimal operation is preferred to keeping a sufficiently high underflow concentration.

If the load instead is more moderate such that the feed point stays within the safe region \( S \) (as in Fig. 6), then there is a possibility to wait with a control action, i.e., to increase \( Q_u \). The SBL rises, but perhaps the higher load is only temporary. Otherwise, a small increase in \( Q_u \) will reduce the speed of the rising SBL. This might be a remedy until the load is reduced again; the smaller the increase in
$Q_u$, the smaller the (undesired) reduction of the underflow concentration. We refer to Diehl (2006) for more such examples.

For step inputs to the underloaded region (from the optimal operation state), the situation is easier. Most commonly, the feed point then belongs to $\Lambda_2$ and it turns out that only one step control action is necessary to maintain optimal operation (a new $Q_u$ is chosen such that the new feed point satisfies $(X_t, s) \in \ell_2(Q_u)$). For details and simulations, see Diehl (2006).

8. Possibilities and limitations for dynamic control

With a full knowledge of step responses and control of such, the next issue concerns a dynamically varying feed point $(X_t(t), s(t))$. It is likely that the properties of step responses are similar to those of the dynamic behaviour. Because of the nonlinear process it is, however, not straightforward to make statements on the dynamic behaviour. The following facts on dynamic solutions of (1) have been proved in Diehl (2006, 2008a):

(i) As long as the SST is in optimal operation, the underflow concentration satisfies

$$X_u(t) \geq \bar{X}_u := \frac{f(X_{\min}, Q_u)}{q_u}. \quad (8)$$

The lower bound $\bar{X}_u$ can be obtained in the control chart, see Fig. 10.

(ii) If a control objective contains the restriction

$$X_u(t) \geq X_u^{\min} \geq \bar{X}_u, \quad (9)$$

where $X_u^{\min}$ is a given minimum bound, then this is fulfilled during dynamic operation if

- the SST is kept in optimal operation, and
- $Q_u(t) \leq Q_u^{\max}$, where $Q_u^{\max}$ is defined as the unique solution of the equation

$$f(X_M(Q_u^{\max}), Q_u^{\max}) = \frac{Q_u^{max}}{A}X_u^{\min}. \quad (10)$$

Note that this maximum bound can be defined a priori. It can be obtained graphically in the control chart, see Fig. 10.

(iii) If the SST is in optimal operation, then this state is left directly when the feed point moves up into the dangerous region $D$ (or $Q_u$ is lowered so that the region $D(Q_u)$ contains the feed point). Two situations may occur: either sludge moves up into the clarification zone, or the concentration just under the feed level becomes higher than $X_{\text{infl}}$.

(iv) If the feed point satisfies $(X_t(t), s(t)) \in S(Q_u(t))$, then, disregarding some exceptional cases, the SST stays in optimal operation until the SBL reaches either the feed level or the bottom.

In the ‘exceptional cases’ in item (4), there may be some sludge in the lower part of the clarification zone during a limited time period. This is probably acceptable in most plants if the clarification zone is not too small. For further details we refer to Diehl (2008a). All in all, using the present model, we recommend that a lower bound on the control variable, $Q_u^{\min}$, is defined continuously such that

$$(X_t(t), s(t)) \in S(Q_u^{\min}(t)) \quad (11)$$

holds.

For a dynamically varying feed point $(X_t(t), s(t))$ a natural direct control strategy would be to set $Q_u(t) := Q_u(X_t(t), s(t))$ continuously. An advantage is that the feed point stays within the safe region $S(Q_u(t))$. Unfortunately, there are several drawbacks. The SBL and the underflow concentration may vary substantially, and (for the activated sludge system) large amplitudes of variation of the recirculation flow to the biological reactor are disadvantageous. Furthermore, for a periodically varying feed point, it turns out that the moving time averages over a period of the total mass and the SBL are not constant, but they decline. This is an effect of the nonlinear system. Consequently, optimal operation is left after some time, see examples in Diehl (2008a).

A refined control strategy with a smaller amplitude of the control variable, and hence smaller amplitudes of the mass and the SBL, is obtained as follows. For simplicity, assume that the feed point is varying periodically between two constant states, a low load and a high load $(X_t(t)$ and $s(t)$ are both square waves), with the mean value point $(X_{0\text{f}}, s_0)$. Firstly, during the time intervals of high load, set $Q_u(t) := Q_u^{\text{min}}(t)$ (see (11)). This is the lowest possible value such that the feed point stays in the safe region. The expectation is that the duration of the high load periods, during which $Q_u^{\min} < Q_u(X_{0\text{f}}, s_0)$ holds, is not so long that the SBL will reach the feed level. Secondly, during the intervals of low load, set $Q_u(t) := Q_u^{\text{low}}$ such that $Q_u^{\text{low}} + Q_u^{\min}/2 = Q_u(X_{0\text{f}}, s_0)$. During intervals of low load, the SBL shrinks back. Then a quasi-stationary solution in optimal operation occurs during some time. The moving time averages of the mass and SBL decline slower than with the direct con-

![Fig. 10. The minimum bound for the underflow concentration $X_u \approx 7.13 \text{kg/m}^3$ satisfies $f(X_{\text{infl}}, Q_u) = \bar{q}_u X_u$. The value $Q_u^{\max}$ can be obtained graphically in the following way. Given $X_u^{\min}$ determine the corresponding $y$-value on the boundary of the sets $\Lambda_3$ and $\Lambda'$. This flux value is equal to $Q_u^{\max} X_u^{\min}/A$.](image-url)
9. Discussion

As the feed point varies with time, it may move from one region to another, which results in a qualitative change in the behaviour of the process. Consider, for example, Figure 2 (right). This is a chart of all possible qualitatively different step responses when the SST initially is in optimal operation in steady state. The complexity of the chart for high feed concentrations is perhaps mostly of theoretical interest. However, for lower feed concentrations in an interval around \( X_m \), there are four main regions (\( U_1, O_1, O_2a, O_3b \)), corresponding to four qualitatively different step responses. This really illustrates the nonlinearity of the model process and the difficulty in controlling it.

Experiments with step responses are scarce in the literature. Those reported by Maljian and Howell (1978) and Laquidara and Keinath (1983) are in qualitative agreement with the ones in Figures 6 and 7. A second-order PDE model, taking the compressibility effects into account, would likely produce solutions closer to experiments. A natural next step in the modelling of the continuous-sedimentation process would be to take the compressibility properties at high concentrations into account. This phenomenon is definitely present for biological sludges. Then one arrives at a second-order model, which is a PDE of the form (1) with an additional nonlinear ‘diffusion term’ that vanishes for concentrations below a critical one, see Bürger et al. (2005); Bürger and Narváez (2007); De Clercq et al. (2008). The adjustments of the operating charts to this second-order model need further research.

10. Conclusions

The solids-flux theory can be described conveniently within a larger theory of nonlinear PDEs with discontinuous coefficients, which has evolved since the 1990s. Concepts such as the feed point (state point), limiting flux, optimal operation, critically, under- or overloaded SST, sludge blanket level and thickening and clarification failure can be identified naturally within a first-order PDE model of the clarification-thickening process. Thereby, a complete description of the stationary behaviour for all loads and comprehensive information on the dynamic behaviour can be provided. Many results are appropriately presented visually by means of operating charts, which enhance the understanding of the nonlinear process and the possibilities for controlling it. The load to the SST is in the operating charts represented by the feed point \((X_f, s)\), which location and movement yield much information.

The main outcomes of the PDE model are the following:
- The steady-state chart in Fig. 2 (left) and the adherent concepts of limiting flux function, critically, under- or overloaded SST, and optimal operation in steady state.
- The graphical construction of all concentrations of steady-state solutions: Fig. 1(d) for a critically loaded SST and Fig. 3 (left/right) for an underloaded/overloaded SST.
- The control chart with respect to steady states in Fig. 4 (right), which is also an important design chart.
- Operating charts with the maximum underflow concentration, maximum surface overflow and maximum thickening factor in Figs 5(a)–(c).
- The operating chart for step responses from optimal operation in Fig. 2 (right).
- The ‘safe’ and ‘dangerous’ regions of the operating chart in Fig. 8(a) concerning the dynamical behaviour. As the feed point stays within the safe region, at worst thickening failure may occur. As the feed point moves into the dangerous region, clarification failure occurs.
- Operating charts with qualitative information on step responses, see Figs 8(b)–(d).

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Wilén, B.-M., Lumley, D., Nordqvist, A., 2004. Dynamics in maximal settling capacity in an activated sludge treat-
Fig. 3. Left: Underloaded SST with \((X_f, s) \in U_1\). The thickening zone can handle all the feed flux; \(f_{thick} = s < f_{lim}(X_f, s)\). Right: Overloaded SST with \((X_f, s) \in O_2\). The upward flux in the clarification zone is equal to the excess flux \(\varepsilon = s - f_{thick}\). A graphical construction of all concentrations is demonstrated. The constant concentration in the clarification zone \(X_{cl}\) can be obtained from the intersection of the graph of \(f\) and the auxiliary line \(y = q_f X - \varepsilon\). The intersection of this line and the underflow line yields \(X_e\). The concentration in the thickening zone is \(X_{th} = X_M\).

Fig. 4. Operating charts for control of steady states. Left: Graphs of \(f_{lim}(\cdot, Q_u) [\text{kg}/(\text{m}^2\text{h})]\) for some values of \(Q_u [\text{m}^3/\text{h}]\). For \(Q_u \geq \bar{Q}_u = 5159 \text{ m}^3/\text{h}\), \(f_{lim} = f\) is strictly increasing. Right: The control chart with respect to steady states. Given a feed point in this chart there is a unique graph \(f_{lim}(\cdot, Q_u)\) that passes through the feed point. With the value \(\bar{Q}_u\) the SST is critically loaded in steady state. Note that this chart is fixed and does not depend on \(Q_u\). It depends only on the batch-settling flux function \(f_b\). The two subsets of \(\Lambda_3 = \Lambda_{3a} \cup \Lambda_{3b}\) are separated by the dashed line at the flux level \(f(X_{infl}, \bar{Q}_u)\).
Fig. 5. Operating charts as the SST is critically loaded and in steady state. Note the dashed curves from the control chart. (a) Contours of the maximum underflow concentration $\tilde{X}_u(X_t, s)$ [kg/m$^3$]. (b) Contours of the maximum surface overflow rate $\tilde{q}_e(X_t, s)$ [m/h]. (c) Contours of the maximum thickening factor $\tilde{\theta}(X_t, s)$ [-].
Fig. 6. A numerical simulation of a step response as the feed point jumps up to $(X_1, s) \in O_{2a}$ demonstrating thickening failure. The SBL is initially at the depth 2 m. It reaches the feed level at the time 4.8 h and overflow occurs at 7.3 h.

Fig. 7. Numerical simulation of a step response in the case $(X_1, s) \in O_{2c}$ demonstrating clarification failure. Besides a rising discontinuity in the thickening zone (the previous SBL), there is immediately a rising shock wave in the clarification zone. Overflow occurs already after slightly more than two hours.
Fig. 8. (a) The safe and dangerous regions are separated by the thick curve. (b) After a feed-point step down to \( U \), the SBL will decline from its initial location at \( x = 2 \) m. The labels on the horizontal lines show the time in hours to reach the new steady state, which appears as the SBL meets the bottom at \( x = 4 \) m. (c) A feed-point step to \((X_f, s) \in O_{2a} \cup O_{3a} \subset S\) results in a rising sludge blanket – thickening failure occurs. The labels on the horizontal lines show the time in hours until the SBL reaches the feed level. (d) A feed-point step to \((X_f, s) \in D\) implies that there is directly a rising discontinuity in the clarification zone – clarification failure occurs. Contours of the constant upward speed [m/h] of this discontinuity are shown.
Fig. 9. Control of a highly loaded SST. The initial location of the SBL is at $x = 3.7$ m and at $t = 0$ h there is a step input to $(X_f, s) = (3, 11.5) \in D(Q_u = 3500 \text{ m}^3/\text{h}) \cap \Lambda_2$. 

Contour of $X(x, t)$ [kg/m$^3$]

Concentration $X(x, t)$ [kg/m$^3$]

Underflow concentration [kg/m$^3$]

Mass in SST [tonnes]

$Q_u(t)$ [m$^3$/h]