ON SCALAR CONSERVATION LAWS WITH POINT SOURCE AND DISCONTINUOUS FLUX FUNCTION

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Abstract. The conservation law studied is $\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\Big(F\big(u(x,t),x\big)\Big) = s(t)\delta(x)$, where u is a concentration, s a source, δ the Dirac measure and $F(u,x) = \begin{cases} f(u), & x > 0 \\ g(u), & x < 0 \end{cases}$ the flux function. The special feature of this problem is the discontinuity that appears along the t-axis and the curves of discontinuity that go into and emanate from it. Necessary conditions for the existence of a piecewise smooth solution are given. Under some regularity assumptions sufficient conditions are given enabling construction of piecewise smooth solutions by the method of characteristics. The selection of a unique solution is made by a coupling condition at $x = 0$, which is a generalization of the classical entropy condition and is justified by studying a discretized version of the problem by Godunov's method.

The motivation for studying this problem is that it arises in the modelling of continuous sedimentation of solid particles in a liquid.

Key words. conservation laws, discontinuous flux, point source.

AMS subject classifications. 35A07, 35L65, 35Q80, 35R05.

1. Preliminaries.

1.1. Introduction. This paper is a shortened version of [7], to which we refer for further details.

Let $u(x,t)$ be a scalar function, describing some kind of density, of the space coordinate x and the time coordinate t. It is well known that solutions of the initial value problem for a non-linear scalar conservation law

$$\begin{aligned} &u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $f \in C^2$, even for $u_0 \in C^\infty$ may form discontinuities after a finite time. By interpreting the problem in a weak sense it is possible to define global discontinuous solutions. Uniqueness is guaranteed by an entropy condition. If f is non-convex, the behaviour of the discontinuities is more complicated than in the convex case, see e.g. Ballou [1]. He uses the method of characteristics to construct piecewise smooth solutions for piecewise constant initial data and "admissible" initial data (see definition in [1]). Cheng [4] uses another method to construct solutions for bounded and piecewise monotone initial data. Dafermos [5] has shown that weak solutions of (1.1) generically are piecewise smooth if f has one inflection point.

Motivated by a part of the modelling of continuous sedimentation of solid particles in a liquid, shortly described in Subsection 1.3, we shall study a more general conservation problem with a point source and a discontinuous flux function. The problem will be described in Subsection 1.2. The questions of existence and uniqueness will be analysed in Section 2, which contains the main results: Theorem 2.17 on existence and Theorems 2.18, 2.19 and 2.20 on uniqueness. The solutions are selected by means of a coupling condition, Condition Γ, which generalizes the classical entropy condition (Proposition 2.9). In Section 3 Condition Γ is numerically justified by studying a discretized conservation problem, obtained by a scheme of Godunov

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type. The equivalence between Condition Γ and the so called *viscous profile condition* is analysed in [8]. The stability of these viscous profiles is studied in [9].

A special case of our problem is the Riemann problem with discontinuous flux function, dealt with in Subsection 2.3. In this problem there is no source term and the initial value is simple. The problem has earlier been addressed by Gimse and Risebro [11]. In [12] Gimse and Risebro have, by construction of a sequence of approximate solutions, proved the existence of a solution of the Cauchy problem for a conservation law with discontinuous flux function arising in two-phase flow. They have left the question of uniqueness open. A uniqueness result for this type of equation is given at the end of Subsection 2.5. The presence of a point source causes considerable complications, even if there is no discontinuity in the flux function. Liu [17] studies non-linear resonance when the source also depends on the state variable u. Another related problem is the initial boundary value problem in the sense of Bardos, Le Roux and Nedelec [2]. In that problem a discontinuity is allowed along the boundary as long as it would like to propagate *out from* the domain. Confining our problem to one quadrant (of the x-t-plane) we get an initial value boundary flux problem, see Subsection 2.6. In this problem the flux at the boundary is prescribed and a value at the boundary is allowed to produce a discontinuity if and only if this discontinuity propagates *into* the domain.

1.2. The Problem and Assumptions. Let $s(t)$ be a source situated at $x = 0$, where the flux function $F(u,x)$ is a discontinuous function of x. Given initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, the weak formulation of the problem is

(1.2)
$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + F\varphi_x)\,dx\,dt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\,dx + \int_0^\infty s(t)\varphi(0,t)\,dt = 0, \quad \varphi \in C_0^\infty(\mathbb{R}^2),$$
$$\text{where} \quad F(u,x) = \begin{cases} f(u), & x > 0 \\ g(u), & x < 0. \end{cases}$$

In distribution sense it can be written

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\Big(F\big(u(x,t),x\big)\Big) = s(t)\delta(x), && x \in \mathbb{R},\ t > 0 \\ &u(x,0) = u_0(x), && x \in \mathbb{R}, \end{aligned}$$

where $\delta(x)$ is the Dirac measure. If u is a smooth function except along $x = 0$, then by standard arguments it is easy to show that (1.2) is equivalent to

$$\begin{aligned} &u_t + f(u)_x = 0, && x > 0,\ t > 0 \\ &u_t + g(u)_x = 0, && x < 0,\ t > 0 \\ &f\big(u(0+,t)\big) = g\big(u(0-,t)\big) + s(t), && t > 0 \\ &u(x,0) = u_0(x), && x \in \mathbb{R}. \end{aligned} \tag{1.3}$$

The weak formulation (1.2) can allow a Dirac measure in u. For example if $f \equiv$ constant $= f_0$ and $g \equiv$ constant $= g_0$, then $u(x,t) = u_0(x) + \delta(x)\int_0^t \big(s(\tau) + g_0 - f_0\big)\,d\tau$ is a solution. To avoid this, we shall mean by a *solution* a function u satisfying (1.2). A function u is said to be *piecewise smooth* if it is bounded and C^1 except along a finite number of C^1-curves in every bounded set, such that the left and right limits of u along discontinuity curves exist. Especially we introduce the notation

$$u_\pm(t) = \lim_{\delta \searrow 0} u(\pm\delta, t)$$
$$u^\pm(t) = \lim_{\varepsilon \searrow 0} u_\pm(t+\varepsilon).$$

The order of the limit processes is significant for example when a discontinuity reaches the t-axis or when $s(t)$ is discontinuous. Notice that $u^\pm(t)$ are continuous from the right. A function of one variable is said to be *piecewise monotone* if there are at most a finite number of points on every bounded interval where a shift of monotonicity occurs. We define a *discrete set* of real numbers to be a set that contains at most a finite number of points on every bounded interval.

Assumptions. In this paper the problem (1.2) will not be treated in full generality. In order to motivate a uniqueness condition for the discontinuity along the t-axis, the analysis is restricted to solutions in the class

$$\Sigma = \{u = u(x,t) : u \text{ is piecewise smooth, } u^\pm(t) \text{ are piecewise monotone}\}.$$

The initial value function u_0 is assumed to be piecewise monotone and piecewise smooth. The source function s is assumed to be piecewise monotone, piecewise smooth and continuous from the right. The flux functions $f, g \in C^2$ are assumed to have at most a discrete number of stationary points and the property $|f(u)|, |g(u)| \to \infty$ as $|u| \to \infty$. The last assumption is made to avoid unbounded solutions, see an example in [7]. To be able to construct a solution of problem (1.2), we assume that it is *regular* in a sense defined in Subsection 2.4.

1.3. Physical Motivation. *Continuous sedimentation* of solid particles in a liquid takes place in a clarifier-thickener unit or settler, see Figure 1.1. The one-dimensional x-axis is shown in the figure. The height of the clarification zone is denoted by H, and the depth of the thickening zone by D. At $x = 0$ the settler is fed with suspended solids at a concentration $u_f(t)$ and at a constant flow rate Q_f (volume per unit time). A high concentration of solids is taken out at the underflow, at $x = D$, at a rate Q_u. The effluent flow Q_e, at $x = -H$, is consequently defined by $Q_e = Q_f - Q_u$. It is assumed that these three flows are positive. The cross-sectional area A is assumed to be constant and the concentration u is assumed to be constant on each cross section. We define the the bulk velocities in the thickening and clarification zone as $v = Q_u/A$ and $w = Q_e/A$, with directions shown in Figure 1.1. The feed inlet is modelled by the source function $s(t) = Q_f u_f(t)/A \geq 0$ (mass per unit area and unit time). The standard batch settling flux $\phi(u)$, introduced by Kynch [15] and still used today, is shown in Figure 1.2, where $u_{\max}$ is the maximal packing concentration and u_{infl} is an inflection point. The phenomena at the feed level may be modelled by the equations (1.3) with the flux functions $f(u) = \phi(u) + vu$ and $g(u) = \phi(u) - wu$. The theory of this paper can also be used to predict the effluent concentrations $u_e(t)$ and $u_u(t)$. An analysis of the sedimentation problem is carried out in [6].

Another context where a discontinuous flux function appears is in the modelling of *two-phase flow through one-dimensional porous media*, see Gimse and Risebro [12] and the references therein. The source function is then $s \equiv 0$ and the flux function F may have several discontinuities in the space coordinate. The qualitative behaviour of these discontinuities may be analysed by letting the flux functions f and g in (1.3) have the shapes as shown in Figure 1.2 with one global minimum, $f(0) = g(0)$ and

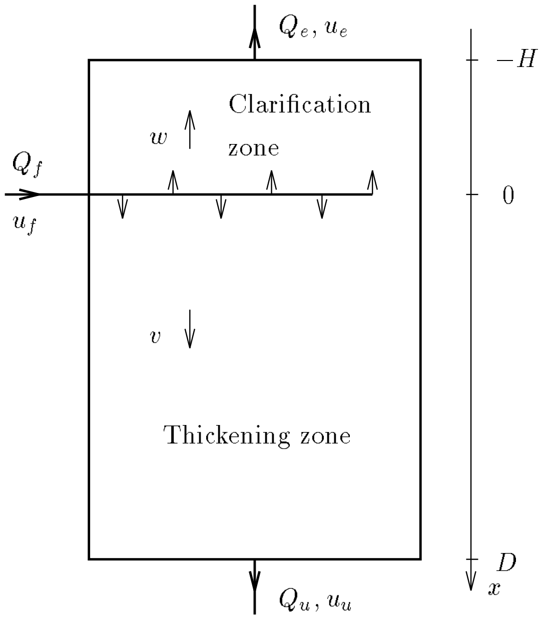

FIG. 1.1. *Schematic picture of the continuous clarifier-thickener.*

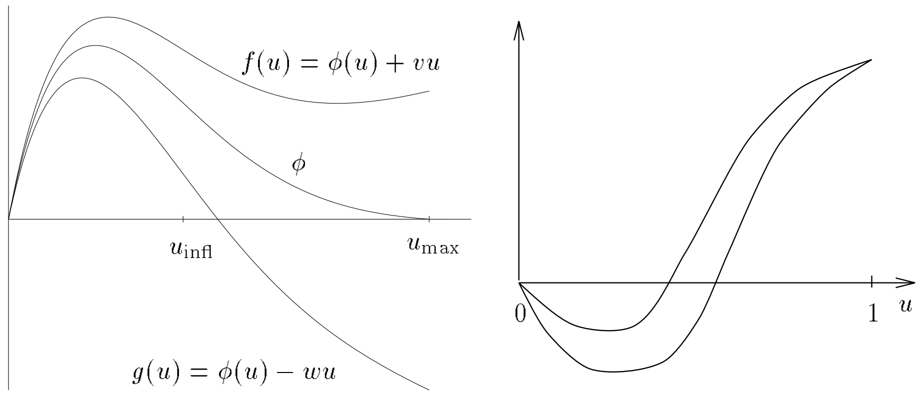

FIG. 1.2. *The flux curves in the sedimentation problem (left) and in the problem of two-phase flow (right). Note that in the right figure either of the graphs can be f or g.*

$f(1) = g(1)$. At the end of Subsection 2.5 are commented the questions of existence and uniqueness for the two problems when f is the upper (lower) and g is the lower (upper) curve in Figure 1.2 (right).

1.4. Properties of Non-Convex Scalar Conservation Laws. In this subsection we review some basic properties of the solution of the scalar problem

$$
\begin{aligned}
&u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t > 0 \\
&u(x,0) = u_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{1.4}
$$

If $x = x(t)$ is a C^1-curve of discontinuity for u, it obeys the *jump condition* or *Rankine-Hugoniot condition*

$$x'(t) = S(u^{x+}, u^{x-}),$$

where $S(\alpha, \beta) = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ for $\alpha \neq \beta$ and $u^{x\pm} = u\big(x(t)\pm 0, t\big)$. Unstable discontinuities are rejected by imposing the *entropy condition*

$$S(v, u^{x-}) \geq S(u^{x+}, u^{x-}), \quad \forall v \text{ between } u^{x-} \text{ and } u^{x+}, \tag{1.5}$$

by Oleinik [18]. Existence and uniqueness of solutions of (1.4) for a general flux f was proved by Kružkov [14]. In the sequel, when talking about solutions of the differential equation $u_t + f(u)_x = 0$ in an open set, the jump condition and the entropy condition are assumed to be fulfilled along curves of discontinuity.

The Riemann problem. The main idea of the analysis of the solution of (1.2) relies heavily on classical results for the solution of the *Riemann problem:*

$$\begin{aligned} &u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0, \end{cases} \end{aligned} \tag{1.6}$$

where u_l and u_r are constants. In the sequel the notation $\mathrm{RP}(f; u_l, u_r)$ will be used for (1.6). The unique solution of (1.6) can be presented as follows. Assume that $u_l < u_r$. For a general flux function f, define $\breve{f}(u) = \breve{f}(u; u_l, u_r)$ on the interval $[u_l, u_r]$ by

$$\breve{f} = \sup\big\{h : h \text{ convex on } [u_l, u_r];\ h(v) \leq f(v),\ \forall v \in [u_l, u_r]\big\}$$

i.e. $\breve{f}$ is the greatest convex minorant of f. The derivative $\breve{f}'$ is a continuous non-decreasing function in the interval (u_l, u_r). Let $\breve{f}'(u_l)$ be interpreted as a right derivative and $\breve{f}'(u_r)$ as a left derivative. Let h denote the inverse function of the restriction of f' to the open intervals where $\breve{f}'$ is increasing. Then the unique weak solution of (1.6), in the case $u_l < u_r$, is

$$u(x,t) = \begin{cases} u_l, & x < \breve{f}'(u_l)t \\ h(\frac{x}{t}), & \breve{f}'(u_l)t < x < \breve{f}'(u_r)t \\ u_r, & x > \breve{f}'(u_r)t \end{cases} \qquad t > 0. \tag{1.7}$$

Inside the cone

$$V(f; u_l, u_r) = \big\{(x,t) : \breve{f}'(u_l)t \leq x \leq \breve{f}'(u_r)t,\ t > 0\big\}$$

the solution consists of rarefaction waves separated by discontinuities. For example if f is concave and $u_l < u_r$, then the cone is merely a straight half line, i.e. a shock. The case when $u_l > u_r$ is treated in the same way, using the least concave majorant instead. If $u_l = u_r$, then the solution is simply $\equiv u_l$ and the cone $V = \emptyset$ by definition.

2. Existence and Uniqueness. In Subsection 2.1 the existence of solutions of (1.2), or (1.3), is characterized in terms of conditions on f, g and s. Away from the t-axis the solution of (1.3) is locally governed by the equations $u_t + f(u)_x = 0$, $x > 0$, and $u_t + g(u)_x = 0$, $x < 0$, separately. To obtain a global solution, we must find boundary functions $\alpha(t)$ and $\beta(t)$ along the t-axis, which together with the initial data define solutions in $x \lessgtr 0$, respectively, such that the fitting of these two solutions defines a global solution $u(x,t)$, satisfying $u^+(t) = \alpha(t)$, $u^-(t) = \beta(t)$ and the third equation of (1.3):

$$f\big(u^+(t)\big) = g\big(u^-(t)\big) + s(t). \tag{2.1}$$

Notice that $s(t)$ is continuous from the right. The key problem is that $\alpha(t)$ and $\beta(t)$ can not be given beforehand in the general case.

The existence of solutions locally in t is proved in Subsection 2.4 by choosing allowable $\alpha(t)$ and $\beta(t)$ such that construction of solutions in $x \lessgtr 0$, respectively, by the method of characteristics is possible.

Non-unique solutions occur when more than one pair of boundary functions (α, β) are allowable. A coupling condition, Condition Γ, is introduced in Subsection 2.2 as a means to pick out a unique solution.

2.1. Definitions and Necessary Conditions. We start with an example showing that there may not exist any solution at all of (1.2).

Example 2.1. Consider the problem when $s \equiv$ constant, $f(u) = -u$, $g(u) = u$ and the initial data $u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$. In this case the characteristics, emanating from the x-axis and carrying the values u_l and u_r, go into the t-axis. Hence $u^-(t) = u_l$ and $u^+(t) = u_r$, $t \geq 0$. A solution of (1.2) exists if and only if $-u_r = u_l + s$, i.e. (2.1) is satisfied.

Suppose the problem (1.2) is solved up to time t, and hence that $u_\pm$ are known. To characterize which $u^\pm$ are possible at time t in order to continue a solution, the following definitions and two lemmas are needed, in which the underlying idea comes from the knowledge of the solution of the Riemann problem (1.6).

Definition 2.2. Given $u_+, u_- \in \mathbb{R}$ and the flux functions f and g, define (see Figures 2.1 and 2.2)

$$\begin{aligned}
\hat{f}(u;u_+) &= \begin{cases} \min\limits_{v\in[u,u_+]} f(v), & u \leq u_+ \\ \max\limits_{v\in[u_+,u]} f(v), & u > u_+ \end{cases}\\
P(f;u_+) &= \{u_+\} \cup \{u : u < u_+;\ \hat{f}(u+\varepsilon;u_+) > \hat{f}(u;u_+),\ \forall\varepsilon > 0\}\\
&\qquad \cup \{u : u > u_+;\ \hat{f}(u-\varepsilon;u_+) < \hat{f}(u;u_+),\ \forall\varepsilon > 0\}\\
\tilde{P}(f;u_+) &= \{u : \hat{f}(u;u_+) = f(u)\}\\
\check{g}(u;u_-) &= \left.\begin{cases} \max\limits_{v\in[u,u_-]} g(v), & u \leq u_- \\ \min\limits_{v\in[u_-,u]} g(v), & u > u_- \end{cases}\right\} = \hat{g}(u_-;u)\\
N(g;u_-) &= \{u_-\} \cup \{u : u < u_-;\ \check{g}(u+\varepsilon;u_-) < \check{g}(u;u_-),\ \forall\varepsilon > 0\}\\
&\qquad \cup \{u : u > u_-;\ \check{g}(u-\varepsilon;u_-) > \check{g}(u;u_-),\ \forall\varepsilon > 0\}\\
\tilde{N}(g;u_-) &= \{u : \check{g}(u;u_-) = g(u)\}.
\end{aligned}$$

Observe that $\hat{f}(\cdot;u_+)$ is a non-decreasing function whose graph consists of increasing parts separated by plateaus, where the function is constant. Analogously

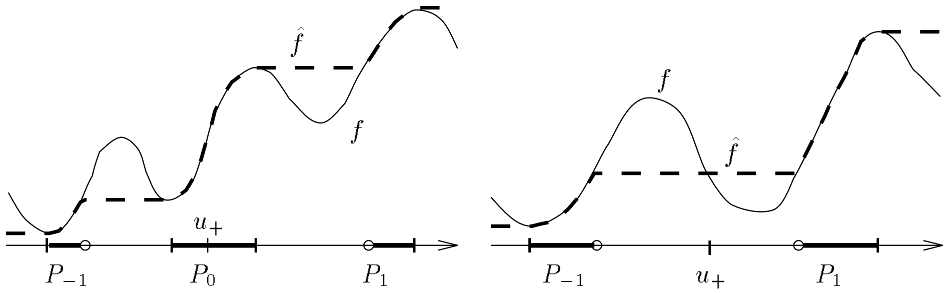

FIG. 2.1. *The graphs of f (solid) and the two main possibilities for $\hat{f}(\cdot; u_+)$ (dashed). The set $P = \bigcup_i P_i$, where $P_0 = \{u_+\}$ in the right hand plot.*

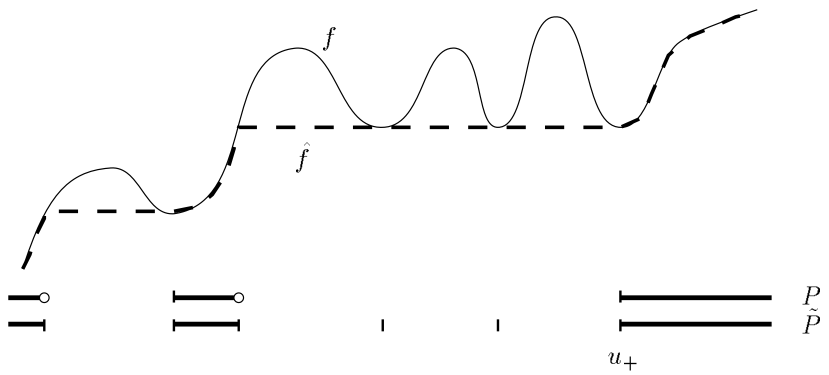

FIG. 2.2. *Examples of the sets P and $\tilde{P}$ for given u_+. Observe that $P \subseteq \tilde{P}$.*

$\breve{g}(\cdot; u_-) = \hat{g}(u_-; \cdot)$ is non-increasing with a graph consisting of decreasing parts separated by plateaus. Furthermore, $\hat{f}$ and $\breve{g}$ are continuous functions in their both variables. In the sequel we shall sometimes use the shorter notation $P = P(f; u_+)$ and $N = N(g; u_-)$. The difference between P and $\tilde{P}$ is illuminated in the Figures 2.1 and 2.2 and in the following lemma. (The proof is found in [7]).

LEMMA 2.3. *Given $u_+, u_- \in \mathbb{R}$, then*

$$\begin{aligned}
P(f; u_+) &= \{\alpha : \text{the solution of } \mathrm{RP}(f; \alpha, u_+) \text{ satisfies } u^+(0) = \alpha\} \\
\tilde{P}(f; u_+) &= \{\alpha : \text{the solution of } \mathrm{RP}(f; \alpha, u_+) \text{ satisfies } u^-(0) = \alpha\} \\
N(g; u_-) &= \{\beta : \text{the solution of } \mathrm{RP}(g; u_-, \beta) \text{ satisfies } u^-(0) = \beta\} \\
\tilde{N}(g; u_-) &= \{\beta : \text{the solution of } \mathrm{RP}(g; u_-, \beta) \text{ satisfies } u^+(0) = \beta\}
\end{aligned}$$

Consider a piecewise smooth solution of (1.2) in the neighbourhood of the t-axis at some time, say $t = 0$. The resolution of a discontinuity approximates the solution of the corresponding Riemann problem, cf. Dafermos [5] and Chang and Hsiao [3]. Since $u_0(x)$ and $u^+(t)$ are smooth for small $x > 0$ and $t > 0$, respectively, the solution u of (1.2) approximates, for small $x > 0$ and $t > 0$, the solution of $\mathrm{RP}\big(f; u^+(0), u_+(0)\big)$ with the cone $V\big(f; u^+(0), u_+(0)\big) \subset \big\{(x,t) : x \geq 0, t > 0\big\}$. An analogous reasoning holds for $x < 0$. These facts yield the following lemma.

LEMMA 2.4. *If u is a piecewise smooth solution of* (1.2) *for $t \in [0, T)$ for some*

$T > 0$, *then*

$$u^+(t) \in \tilde{P}\big(f; u_+(t)\big), \quad t \in [0,T)$$
$$u^-(t) \in \tilde{N}\big(g; u_-(t)\big), \quad t \in [0,T).$$

DEFINITION 2.5. Let t be fixed and $u_+, u_- \in \mathbb{R}$ given. Define the set of intersecting ranges

$$I(u_+, u_-, t) = \hat{f}(\mathbb{R}; u_+) \cap \big(\check{g}(\mathbb{R}; u_-) + s(t)\big).$$

For the projection on the u-axis of the intersection of the graphs of the functions we define

$$\bar{U}(t) = \bar{U}(u_+, u_-, t) = \big\{u \in \mathbb{R} : \hat{f}(u; u_+) = \check{g}(u; u_-) + s(t)\big\},$$
$$\bar{u}_{\max}(t) = \sup \bar{U}(t), \quad \bar{u}_{\min}(t) = \inf \bar{U}(t).$$

Since $\hat{f}$ is non-decreasing and $\check{g}$ is non-increasing, $\bar{U}(t)$ is an interval. When the set $\bar{U}(t)$ only consists of one point, this is denoted by $\bar{u}(t)$. Further introduce the set of pairs

$$\Gamma(u_+, u_-, t) = \big\{(\alpha, \beta) \in \mathbb{R}^2 : f(\alpha) = g(\beta) + s(t) = \hat{f}\big(\bar{U}(t); u_+\big)\big\},$$

see Figure 2.3.

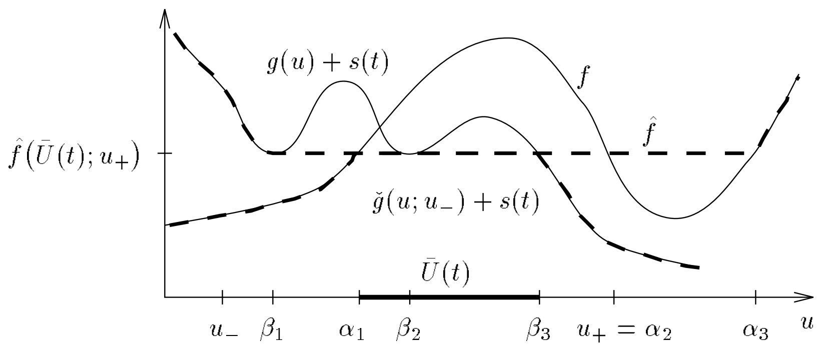

FIG. 2.3. *An example of the set $\bar{U}(t)$. The dashed line from β_1 to β_3 is a plateau of $\check{g}(\cdot; u_-)+s(t)$, and from α_1 to α_3 a plateau of $\hat{f}(\cdot; u_+)$. Note that $\Gamma = \big\{(\alpha_i, \beta_j) : i,j = 1,2,3\big\}$.*

We say that the graphs of $\hat{f}(\cdot; u_+)$ and $\check{g}(\cdot; u_-) + s(t)$ intersect if $I(u_+, u_-, t) \neq \emptyset$. The necessary conditions on the boundary limits $u^{\pm}(t)$ can now be summarized.

THEOREM 2.6 (NECESSARY CONDITIONS). *If u is a piecewise smooth solution of* (1.2) *for $t \in [0,T)$ for some $T > 0$, then*

$$\big(u^+(t), u^-(t)\big) \in \tilde{P}\big(f; u_+(t)\big) \times \tilde{N}\big(g; u_-(t)\big), \quad t \in [0,T)$$
$$\big(u^+(t), u^-(t)\big) \in P\big(f; u_+(t)\big) \times N\big(g; u_-(t)\big), \quad t \in [0,T) \setminus D$$
$$f\big(u^+(t)\big) = g\big(u^-(t)\big) + s(t), \quad t \in [0,T)$$
$$I\big(u^+(t), u^-(t), t\big) \neq \emptyset, \quad t \in [0,T),$$

where D is a discrete set such that

$$D \subseteq \{t_0 \in [0,T) : u^+(t) \text{ or } u^-(t) \text{ is discontinuous at } t = t_0\}.$$

Proof. The first statement is Lemma 2.4. A piecewise smooth solution satisfies $u^+(t) = u_+(t)$ for $t \in [0,T) \setminus D^+$ for some discrete set D^+. For such a t holds by definition $u^+(t) \in P(f; u_+(t))$. Analogously holds $u^-(t) \in N(g; u_-(t))$ for $t \in [0,T) \setminus D^-$ for some discrete set D^-. Letting $D = D^+ \cup D^-$ the second statement is proved, but this can also be true for a set D strictly contained in $D^+ \cup D^-$, see for example the Riemann problem with discontinuous flux function, Section 2.3. The third statement is (2.1). This together with Lemma 2.4 implies $\hat{f}(u^+(t); u_+(t)) = \breve{g}(u^-(t); u_-(t)) + s(t)$ for all $t \in [0,T)$. Since $\hat{f}$ is non-decreasing and $\breve{g}$ is non-increasing there must be an intersection, and the fourth statement is proved. ☐

2.2. The Coupling Condition: Condition Γ. The non-uniqueness of solutions of (1.2) is demonstrated by the following two examples.

EXAMPLE 2.7. Let $s \equiv$ constant, $f(u) = u$, $g(u) = -u$ and the initial data $u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$. Independently of the values $u^\pm$ the characteristics always emanate from the t-axis and hence there exist infinitely many solutions, which satisfy (2.1): $u^+ = -u^- + s$. Note that $P = N = \mathbb{R}$, $\forall t \geq 0$, independently of the initial data. Two possible choices of u^+ and u^- are shown in Figure 2.4, where the jump

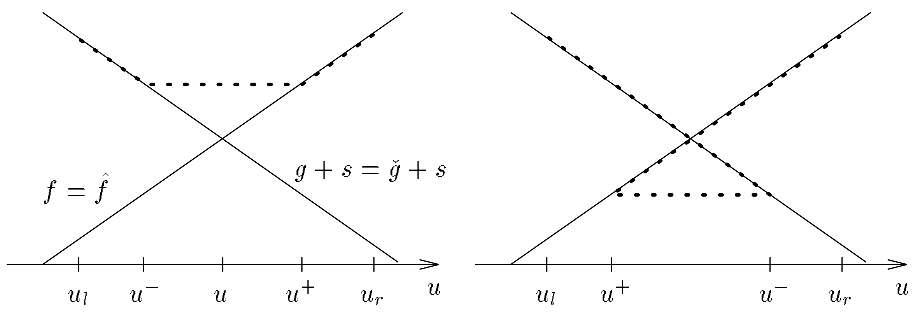

FIG. 2.4. *(Example 2.7) There are infinitely many choices of u^- and u^+.*

from u^- to u^+ corresponds to a horizontal dotted line from the graph of $g(\cdot) + s$ to the graph of f. Of all these solutions the one with $u^- = u^+ = \bar{u}$ turns out to play a distinguished role, see the numerical justification in Theorem 3.1.

EXAMPLE 2.8. Let $s \equiv$ constant, $f(u) = u$, $g(u) =$ a parabola according to Figures 2.5 and 2.6 and the initial data $u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$. Let u_1 and u_2 be the concentrations defined by Figure 2.5, i.e. $g(u_l) + s = f(u_1) = g(u_2) + s$. The solution shown in Figure 2.5 is in accordance with the numerical treatment in Section 3, see Theorem 3.2. Another solution is shown in Figure 2.6.

Gimse and Risebro [11] use the condition to minimize $|u^+ - u^-|$ to obtain a unique solution of the Riemann problem with discontinuous flux function. However, Example 2.8 shows that such a jump need not exist. If interpreting the notation u^+ and u^- in [11] as "inner states" at $x = 0$ instead of as limits of a solution, then a discontinuity along the t-axis (with zero speed) would be allowed with an "inner

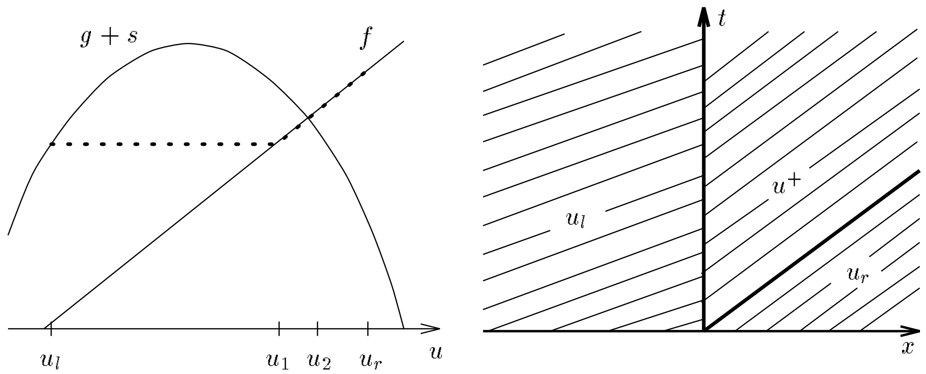

FIG. 2.5. *(Example 2.8) A solution with minimal variation. Notice that* $u^- = u_l$ *and* $u^+ = u_1$*. Thin lines in the right plot are characteristics. The two dotted line segments in the left plot correspond to the two discontinuities in the right plot.*

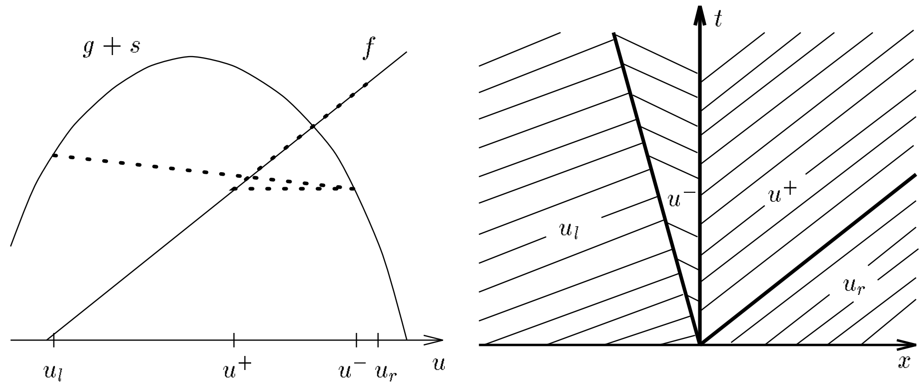

FIG. 2.6. *(Example 2.8) A solution with a smaller jump at* $x = 0$ *than the solution in Figure 2.5, but with greater variation.*

state" on one side of the discontinuity. Then the condition to minimize $|u^+ - u^-|$ can be used as a uniqueness condition. For example, the solution in Figure 2.5 could be seen as the limit of solutions of the type in Figure 2.6 when the speed of the shock in $x < 0$ tends to zero from below.

Since we interpret $u^\pm$ as limits of a solution, the uniqueness condition presented below is based on the intersection of the graphs of $\hat{f}$ and $\breve{g} + s(t)$. Some properties of a unique solution, involving the jump between u^- and u^+ and the variation of the solution, can be found in [7].

In each of the Examples 2.7 and 2.8 a specific solution can be selected by the conservative Godunov scheme of Section 3. Notice that in Example 2.7 holds $\Gamma(u_+, u_-, 0) = \{(\bar{u}, \bar{u})\}$, see Definition 2.5. Letting the pair of boundary functions $\big(\alpha(t), \beta(t)\big) \equiv (\bar{u}, \bar{u})$, $t \geq 0$, then we obtain the same solution as we get by the numerical treatment in Section 3, see Theorem 3.1. In Example 2.8 holds $\Gamma(u_+, u_-, 0) = \big\{(u_1, u_l), (u_1, u_2)\big\}$. The solution obtained by the numerical treatment, see Theorem 3.2, coincides with the analytical solution obtained by using the boundary functions $\big(\alpha(t), \beta(t)\big) \equiv (u_1, u_l)$, $t \geq 0$. In both Examples 2.7 and 2.8 holds $\big(u^+(t), u^-(t)\big) \in \Gamma\big(u_+(t), u_-(t), t\big)$, $\forall t \geq 0$. Motivated by this, as well as by a viscous profile analysis in [8], we introduce the following coupling condition.

CONDITION Γ. For fixed t and given $u_+, u_- \in \mathbb{R}$ holds $(u^+, u^-) \in \Gamma(u_+, u_-, t)$.

Since problem (1.2), or (1.3), is a generalization of (1.1), Condition Γ must be a generalization of the entropy condition.

PROPOSITION 2.9. *If $f \equiv g$ and $s \equiv 0$, then Condition Γ is equivalent to the entropy condition* (1.5).

Proof. Let $x = x(t) \in C^1$ be a discontinuity with u smooth on both sides. By a change of coordinates, under which the entropy condition is invariant, we can assume that the discontinuity has zero speed (replace x by $x - x(t)$ and $f(u)$ by $f(u) - x'(t)u$). Assume that $u^- < u^+$ (the case $u^- > u^+$ is similar).

$$\begin{aligned}
&\text{Condition } \Gamma \iff (u^+, u^-) \in \Gamma(u^+, u^-, t)\\
&\iff f(u^+) = f(u^-) = \hat{f}(\bar{U}; u^+)\\
&\iff f(u^+) = f(u^-) = \hat{f}(u^-; u^+) \quad (\text{since } u^- \in \bar{U})\\
&\iff f(u^+) = f(u^-) = \hat{f}(u^-; u^+) = \min_{u^- \le v \le u^+} f(v)\\
&\iff S(u^-, v) \ge 0 = S(u^-, u^+), \quad \forall v \text{ between } u^- \text{ and } u^+ \qquad \square
\end{aligned}$$

The following example shows (unfortunately) that there exists a time ($t = 0$ in the example) at which $(u^+, u^-) \notin \big(P(f; u_+) \times N(g; u_-)\big)$, i.e. the set D in Theorem 2.6 can be non-empty. It also provides another example of a solution satisfying Condition Γ.

EXAMPLE 2.10. Let $f(u) = u$, $g(u) =$ a parabola according to Figure 2.7, $s(t) = t$ and the initial data $u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$, where u_r is arbitrary and u_l satisfies $f\big(\bar{u}(0)\big) = g\big(\bar{u}(0)\big) = g(u_l)$ according to the figure. Since $s(t)$ is increasing we can let $u^i(t)$ be the smooth increasing function that is the unique intersection of the graphs of $f(\cdot)$ and $g(\cdot) + s(t)$ with $u^i(0) = \bar{u}(0)$. The solution shown in Figure 2.7 satisfies Condition Γ and $u^+(t) = u^-(t) = u^i(t)$ for $t \ge 0$ and in particular $u^-(0) = \bar{u}(0) \in \tilde{N}\big(u_-(0)\big) \setminus N\big(u_-(0)\big)$, where $u_-(0) = u_l$. Note that the

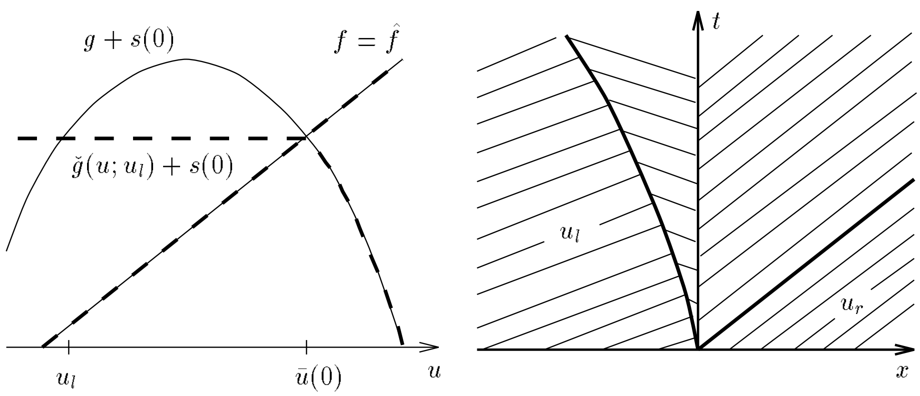

FIG. 2.7. *(Example 2.10) A shock moves to the left separating the value u_l from the values $u^-(t) = u^i(t)$, $t \ge 0$, on the characteristics coming from the t-axis. In $x > 0$ the characteristics emanating from the t-axis carry the values $u^+(t) = u^i(t)$, $t \ge 0$.*

set $\Gamma(u_r, u_l, 0) = \big\{\big(\bar{u}(0), u_l\big), \big(\bar{u}(0), \bar{u}(0)\big)\big\}$ consists of two pairs, but since $s(t)$ is increasing only the pair $\big(\bar{u}(0), \bar{u}(0)\big)$ serves as initial condition for the pair of boundary functions $\big(\alpha(t), \beta(t)\big)$. If $s(t)$ were decreasing, only the other pair of $\Gamma(u_r, u_l, 0)$ would be possible, cf. Example 2.8. This will follow from Theorem 2.19.

2.3. The Riemann Problem with Discontinuous Flux Function. Let the source function s be independent of time. It is no restriction to let $s = 0$. Using the same initial data as in the Riemann problem (1.6), then (1.3) becomes the *Riemann problem with discontinuous flux function*

$$\begin{aligned} & u_t + f(u)_x = 0, && x > 0,\ t > 0 \\ & u_t + g(u)_x = 0, && x < 0,\ t > 0 \\ & f\big(u^+(t)\big) = g\big(u^-(t)\big), && t > 0 \\ & u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0. \end{cases} \end{aligned} \tag{2.2}$$

This problem is treated by Gimse and Risebro [11], cf. the discussion after Example 2.8 above. A solution of this problem can be constructed by a fitting of two "Riemann cones", with $(u^+, u^-) \in P \times N$ according to Lemma 2.3. The following proposition introduces a function $c(u_+, u_-, t) = (u^+, u^-) \in (P \times N) \cap \Gamma$, which will yield the correct boundary values. Furthermore, if u is a solution of (1.2), satisfying Condition Γ, Theorem 2.6 says that $(u^+, u^-) \in (P \times N) \cap \Gamma$ for all t outside a discrete set. Therefore, the function c will also be used in the construction of solutions in the general case, see Subsection 2.4.

PROPOSITION 2.11. *Let t be fixed and $u_+, u_- \in \mathbb{R}$ given. If $I(u_+, u_-, t) \neq \emptyset$, then the set $\big(P(f; u_+) \times N(g; u_-)\big) \cap \Gamma(u_+, u_-, t)$ consists of exactly one point and hence a function c is well defined by*

$$c(u_+, u_-, t) = (u^+, u^-) \in (P \times N) \cap \Gamma.$$

Proof. $I \neq \emptyset$ implies $\bar{U} \neq \emptyset$. Put $\gamma = \hat{f}(\bar{U}; u_+)$. Since the restrictions $f|_P$ and $g|_N$ are injective,

$$(u^+, u^-) \in (P \times N) \cap \Gamma \quad \Longleftrightarrow \quad f|_P\,(u^+) = g|_N\,(u^-) + s(t) = \gamma$$

uniquely determines u^+ and u^-. ☐

We shall now describe the function c by considering all the cases that may occur depending on the set $\bar{U}$:

Case 1. $\bar{u} \in N \cap P$, see Figure 2.8 and Example 2.7. Application of the function c yields $u^+ = u^- = \bar{u}$.

Case 2a. $\bar{u} \in N \setminus P$, see Figure 2.8. The function c yields

$$\begin{aligned} u^+ &= \begin{cases} \max\big(P \cap (-\infty, \bar{u})\big), & u_+ < \bar{u} \\ \min\big(P \cap (\bar{u}, \infty)\big), & u_+ > \bar{u} \end{cases} \\ u^- &= \bar{u}. \end{aligned}$$

Case 2b. $\bar{u} \in P \setminus N$ (symmetric to Case 2a), cf. Example 2.8. The function c yields

$$\begin{aligned} u^+ &= \bar{u} \\ u^- &= \begin{cases} \max\big(N \cap (-\infty, \bar{u})\big), & u_- < \bar{u} \\ \min\big(N \cap (\bar{u}, \infty)\big), & u_- > \bar{u}. \end{cases} \end{aligned}$$

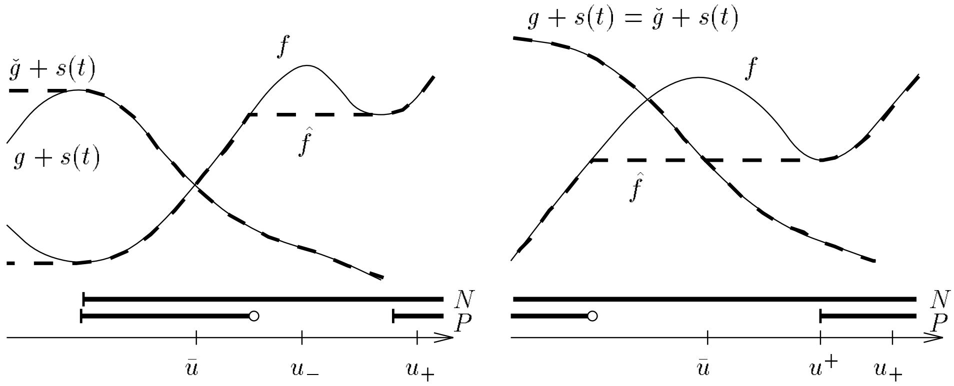

FIG. 2.8. *Examples of Case 1 (left) and Case 2a (right). The sets N and P are indicated above the u-axis.*

Case 3. $\bar{U}$ is infinite or $\bar{u} \in \complement(P \cup N)$, see Figure 2.9. The function c yields

$$u^+ = \begin{cases} \max\left(P \cap (-\infty, \bar{u}_{\min}]\right), & u_+ < \bar{u}_{\min} \\ u_+, & \bar{u}_{\min} \leq u_+ \leq \bar{u}_{\max} \\ \min\left(P \cap [\bar{u}_{\max}, \infty)\right), & u_+ > \bar{u}_{\max} \end{cases}$$

$$u^- = \begin{cases} \max\left(N \cap (-\infty, \bar{u}_{\min}]\right), & u_- < \bar{u}_{\min} \\ u_-, & \bar{u}_{\min} \leq u_- \leq \bar{u}_{\max} \\ \min\left(N \cap [\bar{u}_{\max}, \infty)\right), & u_- > \bar{u}_{\max}. \end{cases}$$

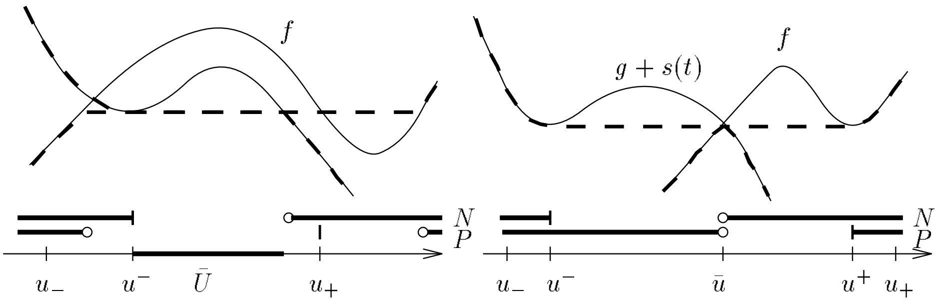

FIG. 2.9. *Examples of Case 3: $\bar{U}$ is infinite (left) and $\bar{u} \in \complement(P \cup N)$ (right). In the left figure holds $u^+ = u_+$.*

THEOREM 2.12. *If $I(u_r, u_l, 0) \neq \emptyset$, then there exists a unique solution $u \in \Sigma$ of (2.2) satisfying Condition Γ for $t \geq 0$. The solution is of the form $u(\frac{x}{t})$ and the constant boundary values are given by $(u^+, u^-) = c(u_r, u_l, 0)$.*

Proof. Let $(\alpha_0, \beta_0) = c(u_r, u_l, 0)$. By Lemma 2.3 there exists a solution, say $v(\frac{x}{t})$, of $\mathrm{RP}(f; \alpha_0, u_r)$ such that the cone $V(f; \alpha_0, u_r)$ is entirely contained in $x \geq 0$, $t > 0$, and $v(0+) = \alpha_0$. Analogously there is a solution $w(\frac{x}{t})$ of $\mathrm{RP}(g; u_l, \beta_0)$ with a cone entirely in $x \leq 0$, $t > 0$, and with $w(0-) = \beta_0$. Since, by definition of c, $f(\alpha_0) = g(\beta_0)$, a function solving the problem is

$$u\left(\frac{x}{t}\right) = \begin{cases} v(\frac{x}{t}), & x > 0 \\ w(\frac{x}{t}), & x < 0 \end{cases} \qquad t > 0.$$

To prove the uniqueness let $\tilde{u}(x,t) \in \Sigma$ be any solution of (2.2) that satisfies Condition Γ $\forall t \geq 0$. First we show that $\big(\tilde{u}^+(t), \tilde{u}^-(t)\big) = (\alpha_0, \beta_0)$ for small $t > 0$. Assume that $\tilde{u}^+(t)$ is non-constant for small $t > 0$. Since $\tilde{u}^+(t)$ is smooth and monotone for small $t > 0$ and f non-constant on every open interval, two cases may appear:

1. $f'\big(\tilde{u}^+(t)\big) < 0$ for small $t > 0$. Then the characteristics to the right of the t-axis have negative speed and must therefore come from the positive x-axis. Hence $\tilde{u}^+(t) \equiv u_r$ for small $t > 0$, which is a contradiction.

2. $f'\big(\tilde{u}^+(t)\big) > 0$ for small $t > 0$. Since $\tilde{u}^+(t)$ is non-constant and g is non-constant on every open interval, the relation (2.1): $f\big(\tilde{u}^+(t)\big) = g\big(\tilde{u}^-(t)\big)$ implies that $\tilde{u}^-(t)$ is non-constant for small $t > 0$. Thus $g\big(\tilde{u}^-(t)\big) = \breve{g}\big(\tilde{u}^-(t); \tilde{u}_-(t)\big)$ is also non-constant and $\tilde{u}^-(t) \in N\big(\tilde{u}_-(t)\big)$ implies that this occurs only if $g'\big(\tilde{u}^-(t)\big) > 0$ for small $t > 0$. Then the characteristics to the left of the t-axis have positive speed and must come from the negative x-axis, carrying the constant value $\tilde{u}^-(t) \equiv u_l$, which is also a contradiction.

Thus $\tilde{u}^+(t) \equiv$ constant for small $t > 0$ must hold and then

$$\tilde{u}(x,t) = \tilde{u}^{\mathrm{RP}}\left(\frac{x}{t}\right) \quad \text{for } x > 0 \text{ and small } t > 0, \tag{2.3}$$

where $\tilde{u}^{\mathrm{RP}}$ is the solution of $\mathrm{RP}\big(f; \tilde{u}^+(0), u_r\big)$. Theorem 2.6 and Condition Γ say that $\big(\tilde{u}^+(0), \tilde{u}^-(0)\big) \in \big(\tilde{P}(u_r) \times \tilde{N}(u_l)\big) \cap \Gamma(u_r, u_l, 0)$, which implies that $\tilde{u}^+(0) = \tilde{u}^{\mathrm{RP}}(0+) = \alpha_0$, and hence $\tilde{u}^+(t) \equiv \alpha_0$ for small $t > 0$. Analogously we conclude that $\tilde{u}^-(t) \equiv \beta_0$ for small $t > 0$. The only possibility left for $\tilde{u}(x,t)$ to differ from $u(\frac{x}{t})$ is that $\tilde{u}^+(t) \neq \alpha_0$ and/or $\tilde{u}^-(t) \neq \beta_0$ for $t > t_0 \in (0,\infty)$. Then the "new initial data" are $\tilde{u}(x,t_0) = u(\frac{x}{t_0}) = v(\frac{x}{t_0})$, $x > 0$. Either $\tilde{u}(x,t_0) \equiv u_r$, $x > 0$, then $\alpha_0 = u_r$, and the reasoning above (at $t = 0$) gives $\tilde{u}^+(t) \equiv u_r = \alpha_0 = u^+(t)$ for small $t - t_0 > 0$. Otherwise $v(\frac{x}{t})$ consists of a cone $V(f; \alpha_0, u_r)$ entirely contained in $x \geq 0$, so that $f'\big(v(\frac{x}{t})\big) > 0$ for small $x > 0$. Almost the same reasoning as above (at $t = 0$) can be used: If $\tilde{u}^+(t)$ is non-constant for small $t - t_0$, then only item 2 is possible and the same reasoning holds and gives a contradiction. Thus $\tilde{u}^+(t) \equiv$ constant for $0 < t < t_0 + \varepsilon$ for some $\varepsilon > 0$, and this implies that (2.3) holds for $0 < t < t_0 + \varepsilon$, so the limit value is again $\tilde{u}^+(t) \equiv \alpha_0$ for $0 < t < t_0 + \varepsilon$, and we have proved that $\tilde{u}^+(t) \equiv u^+(t)$ for $0 < t < t_0 + \varepsilon$. Analogously holds $\tilde{u}^-(t) \equiv u^-(t)$ for small $t - t_0 > 0$ and hence $\tilde{u}(x,t) \equiv u(\frac{x}{t})$ for small $t - t_0 > 0$, which is a contradiction. □

2.4. Construction of Solutions in the General Case. In this subsection the existence of a solution of (1.2) locally in t is proved by construction. When constructing solutions of the simpler problem (1.1), certain assumptions have to be laid on the initial data, see Ballou [1] and Cheng [4]. To construct a solution of (1.2) we must define boundary functions $\alpha(t)$ and $\beta(t)$ with the same regularity as we require of the initial data u_0, i.e. piecewise smoothness and piecewise monotonicity. However, since the main problem of the construction is to define these boundary functions we must impose restrictions on u_0, f, g and s to ensure that $\alpha(t)$ and $\beta(t)$ become piecewise smooth and piecewise monotone. Since the behaviour of a solution changes abruptly when a discontinuity reaches the t-axis, it is natural to formulate conditions for existence locally in t. In Definition 2.14 below, restrictions on u_0, f, g and s are given, defining what we call a *regular* problem. If the problem (1.2) is regular, then we show how a solution $u \in \Sigma$ satisfying Condition Γ can be constructed by the method of characteristics for $0 \leq t < \varepsilon$ for some $\varepsilon > 0$. Then if $u(x, \varepsilon - 0)$ serves as initial data for a new regular problem, starting at $t = \varepsilon$, the solution can be continued. Definition 2.14 is further commented at the end of this subsection.

Below we shall present a "procedure of construction" of a solution and postpone, until Theorem 2.17, the proof that it works and that the solution satisfies Condition Γ as well as that it belongs to the class Σ. Also, the proof of Theorem 2.17 will clarify the steps of the procedure. The idea is the following. From Theorem 2.6 we know that a piecewise smooth solution satisfying Condition Γ fulfils $(u^+, u^-) = c(u_+, u_-, t)$ for all t outside a discrete set. In contrast to the simpler problem in the previous subsection, the function c defined in Proposition 2.11 will be used twice to define the boundary functions $\alpha(t)$ and $\beta(t)$ in the general case. This is due to the dependence of $u_0(x)$ on x and the dependence of $s(t)$ on t. At $t = 0$ the function c is used the first time to define two constants a and b, which are used in two auxiliary problems. These problems produce two functions $\tilde{v}^+(t)$ and $\tilde{w}^-(t)$, on which the function c is applied again to finally define $\alpha(t)$ and $\beta(t)$.

It is convenient to divide the initial data

$$u(x,0) = \begin{cases} u_l(x), & x < 0 \\ u_r(x), & x > 0, \end{cases}$$

and define $u_l(0) \equiv u_l(0-)$ and $u_r(0) \equiv u_r(0+)$.

Procedure of construction.

1. Let $(a, b) = c\big(u_r(0), u_l(0), 0\big)$.
2. Solve the initial value problems

$$\text{(2.4)}\qquad \begin{aligned} &\tilde{v}_t + f(\tilde{v})_x = 0 \\ &\tilde{v}(x,0) = \begin{cases} a, & x < 0 \\ u_r(x), & x > 0 \end{cases} \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\tilde{w}_t + g(\tilde{w})_x = 0 \\ &\tilde{w}(x,0) = \begin{cases} u_l(x), & x < 0 \\ b, & x > 0 \end{cases} \end{aligned}$$

and compute $\tilde{v}^+(t)$ and $\tilde{w}^-(t)$ for $t \geq 0$.

3. Let

$$\text{(2.5)}\qquad T = \sup\big\{t_1 : I\big(\tilde{v}^+(t), \tilde{w}^-(t), t\big) \neq \emptyset, \forall t \in [0, t_1]\big\}$$

and define

$$\text{(2.6)}\qquad \begin{aligned} \big(\alpha(t), \beta(t)\big) &= c\big(\tilde{v}^+(t), \tilde{w}^-(t), t\big), \quad 0 < t < T \\ \big(\alpha(0), \beta(0)\big) &= \big(\alpha(0+), \beta(0+)\big). \end{aligned}$$

4. Solve the initial boundary value problems with $\varepsilon \leq T$ as large as possible, i.e., ε is the first time when $\alpha(t)$ or $\beta(t)$ is discontinuous or when a discontinuity reaches the t-axis,

$$\text{(2.7)}\qquad \begin{aligned} &v_t + f(v)_x = 0, && x > 0,\ 0 < t < \varepsilon \\ &v(x,0) = u_r(x), && x > 0 \\ &v^+(t) = \alpha(t), && 0 \leq t < \varepsilon \end{aligned}$$

and

$$\text{(2.8)}\qquad \begin{aligned} &w_t + g(w)_x = 0, && x < 0,\ 0 < t < \varepsilon \\ &w(x,0) = u_l(x), && x < 0 \\ &w^-(t) = \beta(t), && 0 \leq t < \varepsilon. \end{aligned}$$

5. Let

$$u(x,t) = \begin{cases} v(x,t), & x > 0 \\ w(x,t), & x < 0 \end{cases} \qquad 0 \le t < \varepsilon. \tag{2.9}$$

In step 3 we define $\alpha(t)$ and $\beta(t)$ to be continuous from the right, since they should satisfy $\alpha(t) = u^+(t)$ and $\beta(t) = u^-(t)$ eventually.

EXAMPLE 2.13. If we apply the procedure to the problem in Example 2.10 we obtain 1: $(a,b) = \big(\bar{u}(0), u_l\big)$, 2: $\tilde{v}^+(t) \equiv u_l$ and $\tilde{w}^-(t) \equiv \bar{u}(0)$, 3: $\alpha(t) = \beta(t) = u^i(t)$, $t \ge 0$. Notice that in this example $\big(\alpha(0), \beta(0)\big) \ne c\big(\tilde{v}^+(0), \tilde{w}^-(0), 0\big)$, which motivates the definition (2.6) at $t = 0$. Further we can let $\varepsilon = \infty$ in 4 and the solution $u(x,t)$ in 5 is shown in Figure 2.7.

DEFINITION 2.14. The problem (1.2) is said to be *regular* if the following holds:

1. The solutions $\tilde{v}$ and $\tilde{w}$ of the initial value problems (2.4) belong to Σ for small $t > 0$.
2. The function

$$\eta(t) \equiv f\big(\tilde{v}^+(t)\big) - g\big(\tilde{w}^-(t)\big) - s(t) \tag{2.10}$$

is either $>$, $<$ or $\equiv 0$ on some interval $0 < t < \delta$.

3. If $u^i(t)$ is a unique intersection of $f(\cdot)$ and $g(\cdot) + s(t)$ for small $t > 0$ with $u^i(0) = \bar{u}(0)$, $\bar{u}_{\min}(0)$ or $\bar{u}_{\max}(0)$, then the functions

$$f\big(u^i(t)\big) - f\big(\tilde{v}^+(t)\big),$$
$$g\big(u^i(t)\big) - g\big(\tilde{w}^-(t)\big)$$

are either $>$, $<$ or $\equiv 0$ on some interval $0 < t < \delta$.

Concerning item 1, note that $\tilde{v}$ and $\tilde{w}$ are generically piecewise smooth in the case of one inflection point of f and g, respectively, see Dafermos [5]. This is the case in the applications to sedimentation and two-phase flow, see Figure 1.2. Furthermore, the solutions $\tilde{v}$ and $\tilde{w}$ near the origin are solutions of perturbed Riemann problems with the initial data monotone on each side of $x = 0$. Considering all cases of flux functions, see Chang and Hsiao [3], it follows that $\tilde{v}^+(t)$ and $\tilde{w}^-(t)$ are monotone for small $t \ge 0$.

Before stating the existence theorem, we need two lemmas on some properties of the function c. The first says among others what happens when applying c twice.

LEMMA 2.15. *Let* $\mathrm{ch}(A,B)$ *denote the convex hull of* $A \cup B$, *where* A *and* B *are points or intervals of* $\mathbb{R}$. *Given a fixed* $t \ge 0$ *and* $u_+, u_- \in \mathbb{R}$, *let* $(\alpha_0, \beta_0) = c(u_+, u_-, t)$ *and* $\bar{U}_0 = \bar{U}(u_+, u_-, t)$. *Then*

$$\begin{cases} \alpha \in \mathrm{ch}(u_+, \alpha_0) \\ \beta \in \mathrm{ch}(u_-, \beta_0) \end{cases} \implies \begin{cases} \hat{f}(u;\alpha) = \hat{f}(u;u_+), \ \forall u \in \mathrm{ch}(\bar{U}_0, \alpha_0) \\ \breve{g}(u;\beta) = \breve{g}(u;u_-), \ \forall u \in \mathrm{ch}(\bar{U}_0, \beta_0) \\ \bar{U}(\alpha,\beta,t) = \bar{U}_0 \\ c(\alpha,\beta,t) = (\alpha_0, \beta_0). \end{cases}$$

Proof. Let $\bar{U}_0 = \bar{U}(u_+, u_-, t)$. From Cases 1–3 after Proposition 2.11 it follows that either $\alpha_0 = u_+$ or α_0 lies closer to $\bar{U}_0$ than u_+ and on the same side of $\bar{U}_0$ as u_+. Analogously, either $\beta_0 = u_-$ or β_0 lies closer to $\bar{U}_0$ than u_- and on the same side of $\bar{U}_0$ as u_-. Then Definition 2.2 of $\hat{f}$, P, etc. implies the statements in turn. □

LEMMA 2.16. *Let* $\bar{U}(0) = \bar{U}\big(u_r(0), u_l(0), 0\big)$. *The solutions* $\tilde{v}$ *and* $\tilde{w}$ *of* (2.4) *satisfy*

$$\begin{aligned} &\bar{U}\big(\tilde{v}^+(0), \tilde{w}^-(0), 0\big) = \bar{U}(0) \\ &\hat{f}\big(\bar{U}(0); \tilde{v}^+(0)\big) = \hat{f}\big(\bar{U}(0); u_r(0)\big) \\ &\breve{g}\big(\bar{U}(0); \tilde{w}^-(0)\big) = \breve{g}\big(\bar{U}(0); u_l(0)\big). \end{aligned}$$

Proof. The resolution of the discontinuity between a and $u_r(0)$ is approximated by the solution of $\mathrm{RP}\big(f; a, u_r(0)\big)$, cf. the discussion preceding Lemma 2.4. This implies $\tilde{v}^+(0) \in \mathrm{ch}\big(a, u_r(0)\big)$ and $\tilde{w}^-(0) \in \mathrm{ch}\big(b, u_l(0)\big)$. The statements are implied by Lemma 2.15. □

THEOREM 2.17 (EXISTENCE). *If* (1.2) *is a regular problem and*

$$I(\alpha, \beta, t) \neq \emptyset, \quad \forall (\alpha, \beta, t) \in \mathbb{R} \times \mathbb{R} \times [0, T) \text{ for some } T > 0, \tag{2.11}$$

then there exists a solution $u \in \Sigma$ *satisfying Condition* Γ *for* $t \in [0, \varepsilon)$ *for some* $\varepsilon \in (0, T]$.

REMARK. (2.11) can be replaced by the weaker conditions $I\big(u_r(0), u_l(0), 0\big) \neq \emptyset$ and $T > 0$, with T defined by (2.5).

Proof. Carry out step 1–3 in the "procedure of construction" above. (2.6) gives alone $f\big(\alpha(t)\big) = g\big(\beta(t)\big) + s(t)$ for $t \geq 0$, and together with Lemma 2.15

$$\big(\alpha(t), \beta(t)\big) = c\big(\alpha(t), \beta(t), t\big), \quad t > 0. \tag{2.12}$$

In particular (2.12) means $\big(\alpha(t), \beta(t)\big) \in \Gamma\big(\alpha(t), \beta(t), t\big)$, $t > 0$. It remains to verify Condition Γ at $t = 0$. The continuity of $\hat{f}$ implies that $\hat{f}\left(\bar{U}\big(\tilde{v}^+(t), \tilde{w}^-(t), t\big); \tilde{v}^+(t)\right)$ is a continuous function of t for small $t \geq 0$. Using (2.6), the continuity of $\hat{f}$ and $\breve{g}$ and letting $t \to 0+$ we get

$$f\big(\alpha(0)\big) = g\big(\beta(0)\big) + s(0) = \hat{f}\left(\bar{U}\big(\tilde{v}^+(0), \tilde{w}^-(0), 0\big); \tilde{v}^+(0)\right).$$

This together with Lemma 2.16 gives $\big(\alpha(0), \beta(0)\big) \in \Gamma\big(u_r(0), u_l(0), 0\big)$.

Below it will be shown that $\alpha(t)$ and $\beta(t)$ are smooth and monotone for $0 < t < \varepsilon$ for some $\varepsilon > 0$ (we can assume that $s(t)$ is continuous in this interval). Then (2.12) and Lemma 2.3 ensure that the method of characteristics can be applied to construct solutions v and w in the strip $0 \leq t < \varepsilon$ of the initial boundary value problems (2.7) and (2.8). Then $u(x, t)$ in (2.9) is a solution in Σ of (1.2) for $0 \leq t < \varepsilon$. Near the origin this solution is smooth except along the t-axis and along two possible discontinuities emanating from the origin going into $x > 0$ and $x < 0$, respectively.

By the assumption that $u(x, 0)$ is piecewise smooth and piecewise monotone there exists a $\delta > 0$ such that one of the following alternatives holds:

I. $f'\big(u_r(x)\big) \geq 0$ for $0 < x < \delta$ and $g'\big(u_l(x)\big) \leq 0$ for $-\delta < x < 0$. Then $\big(\tilde{v}^+(t), \tilde{w}^-(t)\big) \equiv (a, b)$ holds, which implies that $\hat{f}\big(\cdot; \tilde{v}^+(t)\big) \equiv \hat{f}(\cdot; a)$ and $\breve{g}\big(\cdot; \tilde{w}^-(t)\big) \equiv \breve{g}(\cdot; b)$ are independent of time. Since by assumption s is monotone for small $t \geq 0$, there exists an $\varepsilon_1 > 0$ such that one of the following cases occurs:

   A. $s(t) \equiv s(0)$, $0 < t < \varepsilon_1$. Then α and β defined by (2.6) are constants. Define v and w by (2.7) and (2.8). Let $\varepsilon \in (0, \varepsilon_1]$ be the first time a discontinuity crosses the t-axis. Then (2.9) defines a solution for $0 < t < \varepsilon$.

B. $s(t) \neq s(0)$, $0 < t < \varepsilon_1$. Then (2.6) implies that α and β satisfy

$$\hat{f}\big(\alpha(t); a\big) = \check{g}\big(\beta(t); b\big) + s(t), \quad t > 0. \tag{2.13}$$

The graph of the non-decreasing function $\hat{f}$ consists of increasing parts separated by plateaus (where $\hat{f} \equiv$ constant) and analogously the graph of $\check{g}$ consists of decreasing parts separated by plateaus. Define the set $\bar{U}(t) = \bar{U}(a, b, t)$, $t \geq 0$. Three cases may occur:

(i) There is a unique intersection at $\bar{u}(0)$, with $f'(u) = \hat{f}'(u) > 0$ and $g'(u) = \check{g}'(u) < 0$ $\forall u$ $(\neq \bar{u}(0))$ in a neighbourhood of $\bar{u}(0)$. The parenthesis in the previous sentence applies if $\bar{u}(0)$ happens to be an inflection point. Condition Γ implies that $\alpha(0) = \beta(0) = \bar{u}(0)$ and (2.13) reduces to

$$f\big(\bar{u}(t)\big) = g\big(\bar{u}(t)\big) + s(t),$$

which defines $\alpha(t) = \beta(t) = \bar{u}(t) \in C^1(0, \varepsilon_2)$ for some $\varepsilon_2 \in (0, \varepsilon_1]$ by the implicit function theorem. The monotonicity assumption on $s(t)$ and the fact that f is increasing and g decreasing in a neighbourhood of $\bar{u}(0)$ yield that

$$\bar{u}'(t) = \frac{s'(t)}{f'\big(\bar{u}(t)\big) - g'\big(\bar{u}(t)\big)}$$

is either positive, negative or zero on $0 < t < \varepsilon_3$ for some $\varepsilon_3 \in (0, \varepsilon_2]$. Hence $\alpha(t) = \beta(t) = \bar{u}(t)$ is also monotone for $t \in (0, \varepsilon_3)$. Let (2.7) and (2.8) define solutions v and w and let $\varepsilon \in (0, \varepsilon_3]$ be the first time a discontinuity enters the t-axis. Then (2.9) defines a solution for $0 < t < \varepsilon$.

(ii) There is a unique intersection at $\bar{u}(0)$ which separates a plateau and a strictly monotone part of $\hat{f}$ or $\check{g} + s(0)$. Since $s(t)$ is either increasing or decreasing for small $t > 0$ there is either an intersection as in (i) or an intersection with exactly one plateau involved for small $t > 0$. In the latter case one of α and β is $\equiv$ constant and the other is defined by (2.13) and is smooth and monotone for $0 < t < \varepsilon_2 \in (0, \varepsilon_1]$ by the implicit function theorem and the monotonicity assumption on $s(t)$. Let (2.7) and (2.8) define solutions v and w and let $\varepsilon \in (0, \varepsilon_2]$ be the first time a discontinuity enters the t-axis. Then (2.9) defines a solution for $0 < t < \varepsilon$.

(iii) $\bar{U}(0)$ is infinite, i.e. a plateau of $\hat{f}$ coincides with a plateau of $\check{g} + s(0)$ at $t = 0$. The assumption 2 of Definition 2.14 implies that the plateaus separate immediately and we get a unique intersection as in (i).

II. $f'\big(u_r(x)\big) \geq 0$ for $0 < x < \delta$ and $g'\big(u_l(x)\big) > 0$ for $-\delta < x < 0$. Then $\tilde{v}^+(t) \equiv a$ and hence $\hat{f}$ is independent of time for small $t > 0$. $\tilde{w}^-(t)$ is defined by the characteristics from the negative x-axis carrying the values $u_l(x)$. Then we say that one plateau of the graph of $\check{g}\big(\cdot; \tilde{w}^-(t)\big)$ is *moving* and we denote the set of the corresponding u-values by

$$M(t) = \big\{u : \check{g}\big(u; \tilde{w}^-(t)\big) = g\big(\tilde{w}^-(t)\big)\big\}, \quad t > 0,$$

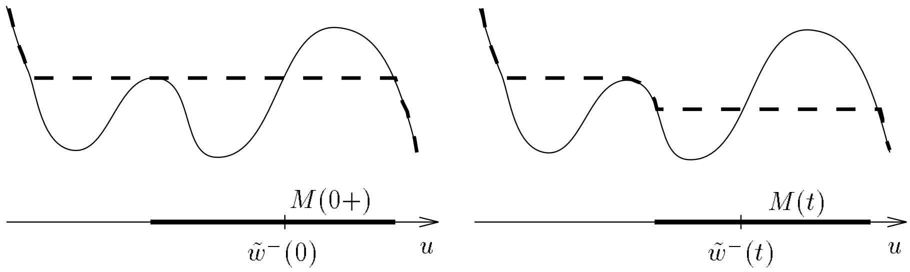

FIG. 2.10. *The moving plateau at $t = 0$ (left) and at some $t > 0$ (right) in a case when $\tilde{w}^-(t)$ is decreasing.*

see Figure 2.10. The other plateaus are called *fixed*. Also define $\bar{U}(t) = \bar{U}\big(a, \tilde{w}^-(t), t\big)$, $t \geq 0$. A solution can be constructed as in case I with some modifications depending on the moving plateau. Instead of making a division depending on $s(t)$ (IA and IB), we must consider the sign of (2.10); $\eta(t) = f(a) - g\big(\tilde{w}^-(t)\big) - s(t)$, which is either positive, negative or zero for small $t > 0$ by the regularity assumption (item 2 of Definition 2.14). Then for example $\eta(t) \equiv 0$ for small $t > 0$ means that the moving plateau lies on the fixed value $f(a)$, but with one or both end points moving smoothly and monotonically (by the implicit function theorem and the assumption on $s(t)$) and hence (2.6) will yield smooth and monotone functions $\alpha(t)$ and $\beta(t)$ for small $t > 0$ as in case I.

One new complication arises when there is a unique intersection at $\bar{u}(0)$, which also is an end point of a moving plateau. Let us study the case when $\bar{u}(0) = \min M(0+)$ in Figure 2.10 and $\tilde{w}^-(t)$ is decreasing. If $s(t)$ is decreasing for small $t > 0$, then there is a unique intersection of the graph of f and the fixed plateau of $\breve{g}$ in Figure 2.10 and a solution is defined as in IB(ii). If $s(t) \geq s(0)$ for small $t > 0$, then there is a unique intersection $u^i(t)$ of the graphs of $f(\cdot)$ and $g(\cdot) + s(t)$ for small $t > 0$ with $u^i(0) = \bar{u}(0)$. Then item 3 of the regularity assumption gives that $g\big(u^i(t)\big) - g\big(\tilde{w}^-(t)\big)$ is either negative or non-negative for small $t > 0$, i.e. either the graph of f intersects the plateau or the decreasing part of $\breve{g}\big(\cdot; \tilde{w}^-(t)\big)$ for small $t > 0$. Hence a solution is defined either as in IB(i) or IB(ii).

III. $f'\big(u_r(x)\big) < 0$ for $0 < x < \delta$ and $g'\big(u_l(x)\big) \leq 0$ for $-\delta < x < 0$. This case is symmetrical to the previous one.

IV. $f'\big(u_r(x)\big) < 0$ for $0 < x < \delta$ and $g'\big(u_l(x)\big) > 0$ for $-\delta < x < 0$. In this case both $\hat{f}$ and $\breve{g}$ have moving plateaus. Because of the assumptions 2 and 3 of Definition 2.14 a solution can be constructed with an extension of cases II and III similar to the extension from case I to case II.

Finally note that the constructed solution belongs to Σ. $\square$

The technical reason for Definition 2.14 is that we want to avoid plateaus (of $\hat{f}$ or $\breve{g}$) oscillating with unbounded frequencies. The restrictions means thus that a plateau either stays fixed or moves monotonically (for small $t > 0$) away from for example another plateau. In each case only one particular pair of the set Γ will be possible as initial value for the pair of boundary functions (α, β), see Subsection 2.5. However, even though three different solutions appear in these cases of movements of a plateau, these three solutions approximate each other for small $t > 0$. This is because whatever pair of Γ is chosen, a possible discontinuity between u_- and u^- (or

u_+ and u^+) will have zero speed for $t = 0+$, cf. the discussion after Example 2.8. This indicates that oscillations of bounded variation would not cause any trouble and it seems plausible that the regularity assumption in Definition 2.14 could be relaxed considerably. However, for the applications we have in mind it is very unlikely that one should find a situation when the problem is not regular, and hence the procedure of construction could be used repeatedly to obtain a global solution.

Notice that letting either $u_l(x)$ and $u_r(x)$ be constant or $s(t)$ be constant does not simplify the construction of solution and proof of existence in this subsection. If all three functions are constant we have the Riemann problem with discontinuous flux function (2.2).

2.5. Proof of Uniqueness. In this subsection we shall outline how to prove that the solution constructed by the method in Subsection 2.4 is the only one in the class Σ that satisfies Condition Γ. The proofs of Theorems 2.18, 2.19 and 2.20 below provide many examples of the construction of a solution. The same notation as in Subsection 2.4 will be used.

Consider the three cases after Proposition 2.11. When there is a unique intersection as in Case 1 (in an open time interval) it is easy to construct a solution satisfying Condition Γ. It will simply satisfy $u^+(t) = u^-(t) = \bar{u}(t)$. This solution is trivially unique since the set $\Gamma\big(u_+(t), u_-(t), t\big) = \Gamma\big(\bar{u}(t), \bar{u}(t), t\big) = \big\{\big(\bar{u}(t), \bar{u}(t)\big)\big\}$ consists of only one pair. When there is an intersection as in Case 2 or 3, there is at least one plateau, of $\hat{f}$ or $\check{g}$, involved, which may imply that the set Γ consists of more than one pair. As we have seen in the proof of Theorem 2.17 this or these plateaus can "move", see Figure 2.10. Depending on whether a plateau, say of $\check{g} + s(t)$, moves up or down, or stays fixed, in relation to for example a plateau of $\hat{f}$, only one pair of the set Γ can be used in each case to obtain a solution. The correct pair is chosen by the construction procedure in Subsection 2.4. The proof of uniqueness consists of excluding all other pairs of Γ. To perform these exclusions we shall use a result by Bardos, Le Roux and Nedelec [2] concerning the two initial boundary value problems

$$\begin{aligned} &v_t + f(v)_x = 0, && x > 0,\ t > 0 \\ &v(x,0) = u_r(x), && x > 0 \\ &v^+(t) \in \tilde{N}\big(f;\alpha(t)\big), && t \geq 0 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &w_t + g(w)_x = 0, && x < 0,\ t > 0 \\ &w(x,0) = u_l(x), && x < 0 \\ &w^-(t) \in \tilde{P}\big(g;\beta(t)\big), && t \geq 0. \end{aligned} \tag{2.15}$$

Observe the arguments f and g of $\tilde{N}$ and $\tilde{P}$. A solution of (2.14) is thus allowed to have a jump at $x = 0$ from $\alpha(t)$ to $v^+(t)$ if this discontinuity would like to move to the left, i.e. if $S\big(\alpha(t), v^+(t)\big) \equiv \frac{f(\alpha) - f(v^+)}{\alpha - v^+} \leq 0$. Dubois and Le Floch [10] introduce the set

$$\mathcal{E}(f;\alpha) \equiv \big\{u \in \mathbb{R} : S(\alpha, k) \leq 0 \text{ for every } k \in \mathrm{ch}(u,\alpha)\big\},$$

and they show the first equality in

$$\mathcal{E}(f;\alpha) = \big\{u^+(0) : u \text{ is the solution of } \mathrm{RP}(f;\alpha,\beta),\ \beta \in \mathbb{R}\big\} = \tilde{N}(f;\alpha),$$

where the last equality is implied by Lemma 2.3. They also show that $\mathcal{E}(f;\alpha)$ is equal to

$$\mathcal{E}_1(f;\alpha) \equiv \left\{ u : \max_{k \in \mathrm{ch}(u,\alpha)} \operatorname{sgn}\left[(u-\alpha)\big(f(u)-f(k)\big)\right] = 0 \right\}.$$

Bardos, Le Roux and Nedelec [2] have shown that there exists a unique solution v of (2.14) that satisfies $v^+(t) \in \mathcal{E}_1\big(f;\alpha(t)\big)$. This is done by a vanishing viscosity approach. We shall use this result, and the symmetrical one for (2.15), to prove that there is only one pair of functions (α,β) that satisfies Condition Γ as well as $v^+(t)=\alpha(t)$, $w^-(t)=\beta(t)$ for small $t \geq 0$. Notice that $\alpha(t)$ and $\beta(t)$ are required to be continuous from the right at $t=0$.

THEOREM 2.18. *If problem* (1.2) *is regular with g increasing, f arbitrary and $I(\alpha,\beta,t) \neq \emptyset$, $\forall(\alpha,\beta,t) \in \mathbb{R}\times\mathbb{R}\times[0,T)$ for some $T>0$, then there exists a unique solution $u \in \Sigma$ for $t \in [0,\varepsilon)$ for some $\varepsilon \in (0,T]$.*

REMARK. Note that Condition Γ is automatically fulfilled since $\breve{g} \equiv$ constant for each t. A similar theorem holds for the symmetrical case when g is arbitrary and f is decreasing.

Proof. We shall only treat some cases here and we refer to [7] for the rest. The existence follows from Theorem 2.17. Let u denote the solution constructed there. Since $g'(u) > 0$, except at a discrete set of inflection points, all characteristics in $x<0$ have positive speed and they define $u^-(t)=\beta(t)$ uniquely. Thus $u^-(0)=u_l(0)$ and hence $\breve{g}\big(\cdot;u^-(t)\big) \equiv g\big(u^-(t)\big)$ for $t \geq 0$. It remains to prove that there is only one possibility to choose the boundary function $\alpha(t)$, namely the one used in the construction in Theorem 2.17. With the notation used there, define $\bar{U}(t) = \bar{U}\big(\tilde{v}^+(t),u^-(t),t\big)$ for small $t \geq 0$. Notice that Lemma 2.16 gives $\bar{U}(0) = \bar{U}\big(u_r(0),u_l(0),0\big)$ and $\hat{f}\big(\bar{U}(0);\tilde{v}^+(0)\big) = \hat{f}\big(\bar{U}(0);u_r(0)\big)$. Two main cases may appear:

1. There is a unique intersection at $\bar{u}(0)$. Hence $\alpha(t)$ is uniquely determined by $\bar{u}(t)=\alpha(t)$ for small $t>0$.

2. The set $\bar{U}(0)$ is infinite, i.e. a plateau of $\hat{f}$ coincides with the constant $g\big(u^-(0)\big)+s(0)$. Depending on the graph of f there are two main cases: A. $f\big(u_r(0)\big) \neq \hat{f}\big(\bar{U}(0);\tilde{v}^+(0)\big)$ and B. $f\big(u_r(0)\big) = \hat{f}\big(\bar{U}(0);\tilde{v}^+(0)\big)$. We shall only treat case A here and refer to [7] concerning all subcases that occur in case B. By symmetry it suffices to assume that $u_r(0) > \bar{u}_{\max}(0)$, see Figure 2.11. Let $\alpha_i, \in \{1,\dots,n\}$ be all

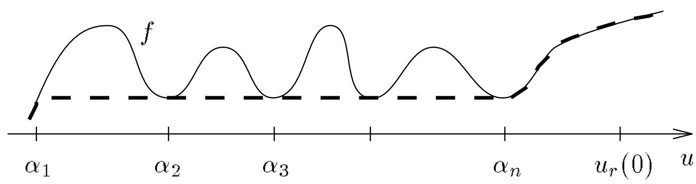

FIG. 2.11. *Case 2A. The dashed graph is* $\hat{f}\big(\cdot;\tilde{v}^+(0)\big) = \hat{f}(\cdot;\alpha_n)$.

possible u-values that satisfy $f(\alpha_i) = \hat{f}\big(\bar{U}(0);\tilde{v}^+(0)\big)$ numbered according to Figure 2.11. Notice that $\hat{f}\big(\bar{U}(0);u_r(0)\big) = \hat{f}\big(\bar{U}(0);\alpha_n\big)$. The conservation law (2.1): $f\big(\alpha(t)\big) = g\big(u^-(t)\big)+s(t)$ implies that for every admissible $\alpha(t)$ must hold $\alpha(0)=\alpha_i$ for some $i \in \{1,\dots,n\}$. Since $\tilde{v}^+(t) \equiv a = \alpha_n$ for small $t \geq 0$, which gives $f\big(\tilde{v}^+(t)\big) \equiv f(\alpha_n) = \hat{f}\big(\bar{U}(0);u_r(0)\big)$ for small $t \geq 0$, (2.10) becomes $\eta(t) = f(\alpha_n) - g\big(u^-(t)\big) - s(t)$. By the regularity assumption three cases may occur:

(i) $\eta(t) \equiv 0$ for small $t > 0$. Then $\alpha(t) \equiv \alpha_i$ for small $t > 0$, for some i. Independently of i the unique solution v of (2.14) satisfies $v(x,t) = u^{\mathrm{RP}}(\frac{x}{t})$ in $\{(x,t) : 0 < x < Kt,\ t > 0\}$ for some $K > 0$, where u^{RP} is the solution of $\mathrm{RP}(f;\alpha_i,u_r(0))$, for $v^+(t) = u^{\mathrm{RP}}(0+) \in \tilde{N}(f;\alpha(t))$. Thus uniquely $v^+(t) = u^{\mathrm{RP}}(0+) = \alpha_n = u^+(t)$ for small $t > 0$.

(ii) $\eta(t) > 0$ for small $t > 0$. Independently of α_i there is a unique intersection as in 1 with $u^+(0) = \alpha_1$.

(iii) $\eta(t) < 0$ for small $t > 0$. According to the construction $\alpha(0) = \alpha_n$ and $\alpha(t) > \alpha_n$ for small $t > 0$. The solution is defined as in case 1. Suppose instead that the boundary function $\alpha(t)$ satisfies $\alpha(0) = \alpha_i$ for some $i \in \{1,\dots,n\}$ with $\alpha(t) < \alpha_n$ and $f(\alpha(t)) = g(u^-(t)) + s(t) > f(\alpha_n)$ for small $t > 0$. Then the solution v of (2.14) satisfies $v^+(t) = \alpha_n \neq \alpha(t)$. Hence the solution constructed in Theorem 2.17 is the only possible one. □

THEOREM 2.19. *If problem* (1.2) *is regular with g decreasing, f arbitrary and $I(\alpha,\beta,t) \neq \emptyset$, $\forall(\alpha,\beta,t) \in \mathbb{R}\times\mathbb{R}\times[0,T)$ for some $T > 0$, then there exists a unique solution $u \in \Sigma$ satisfying Condition Γ for $t \in [0,\varepsilon)$ for some $\varepsilon \in (0,T]$.*

A similar theorem holds for the symmetrical case when g is arbitrary and f is increasing. The proof is found in [7].

Notice that Theorem 2.18 deals with the case of intersection when $\breve{g}$ is one plateau from $-\infty$ to ∞ for every t and Theorem 2.19 the case when $\breve{g}$ is decreasing for every t. The following theorem includes the case when a plateau with moving end point(s) is involved in the intersection. The proof is found in [7].

THEOREM 2.20. *If problem* (1.2) *is regular with $f \equiv g$ having precisely one stationary point, which is a global minimizer $u_{\min}$, then there exists a unique solution $u \in \Sigma$ satisfying Condition Γ for $t \in [0,\varepsilon)$ for some $\varepsilon > 0$.*

This theorem gives that if for example f is convex, then there always exists a unique solution of the initial value problem for the equation $u_t + f(u)_x = s(t)\delta(x)$ (assuming regularity).

For the flux functions in the problem of continuous sedimentation, see Figure 1.2, uniqueness in the class Σ is shown in [6].

Recall the problem of two-phase flow in porous media, Subsection 1.3. The flux functions f and g in Figure 1.2 (right) have precisely one stationary point, which is a global minimizer. This is qualitatively the same as in Theorem 2.20 with the simplification that the source function is $s \equiv 0$. There are two qualitatively different possibilities for $\hat{f}$ and two for $\breve{g}$, depending on whether $u_\pm$ lie to the left or to the right of the minimum. As a simpler case than the sedimentation problem, see [6], it can be proved that $0 \le u_0(x) \le 1 \Rightarrow 0 \le u(x,t) \le 1$ and $I(\alpha,\beta,t) \neq \emptyset$, $\forall(\alpha,\beta,t) \in [0,1]\times[0,1]\times[0,\infty)$. Hence the procedure of construction in Subsection 2.4 can be applied if the problem is regular. Gimse and Risebro [12] have proved the existence of a global solution if the initial value u_0 has bounded total variation. They have left the question of uniqueness as an unsolved problem. The proof of Theorem 2.20 yields uniqueness in the class Σ.

2.6. The Initial Value Boundary Flux Problem.

Consider the *initial value boundary flux problem*

$$
\begin{aligned}
&u_t + f(u)_x = 0, && x > 0,\ t > 0\\
&u(x,0) = u_0(x), && x > 0 \qquad (2.16)\\
&f(u^+(t)) = f_0(t), && t \ge 0.
\end{aligned}
$$

This is a variant of (1.3) when $g \equiv 0$ and $s(t) = f_0(t)$. The definitions and results above may be modified to this problem. For example in the analogue of Theorem 2.6 no restrictions are laid on u_-. The construction of a solution can be done as in Subsection 2.4 with obvious modifications. The analysis now relies on the intersection of the graph of $\hat{f}(\cdot; u_+(t))$ and the constant graph $f_0(t)$. Hence the proof of Theorem 2.18 also yields the following theorem. Note that Condition Γ is automatically fulfilled.

THEOREM 2.21. *If* (2.16) *is regular and* $f_0(t) \in \hat{f}(\mathbb{R}; \alpha)$, $\forall(\alpha, t) \in \mathbb{R} \times [0, T)$ *for some* $T > 0$, *then there exists a unique solution* $u \in \Sigma$ *for* $t \in [0, \varepsilon)$ *for some* $\varepsilon \in (0, T]$.

3. Justification of Condition Γ by Godunov's Method. In this section we shall justify Condition Γ by studying a discretized version of our conservation law problem (1.2). The idea of Godunov's [13] numerical method for an equation $u_t + f(u)_x = 0$ is to use the integral form of the conservation law and the entropy solution of the Riemann problem (1.6) to form an approximate solution by means of a discretization. The extension of this procedure to our problem (1.2), which includes the source along the t-axis, is straightforward if placing grid points *on* the t-axis. The scheme is presented in Subsection 3.1.

It is well known that Godunov's method for a scalar equation $u_t + f(u)_x = 0$ produces a sequence of approximate solutions that converges to the unique entropy satisfying solution provided such solutions of the Riemann problem (1.6) are used in the derivation of the algorithm, see Le Roux [19]. The extension of Godunov's method to our problem does not include any extra condition along the t-axis, so it is suitable for a justification of Condition Γ. No convergence proof of the algorithm is presented here.

In Subsection 3.2 the extension of Godunov's method is used on the Riemann problem with discontinuous flux function (2.2) in two cases.

3.1. Extension of Godunov's Method to Problem (1.2). For δ and $\tau > 0$ let $\{(i\delta, j\tau) : i, j \in \mathbb{Z},\ j \geq 0\}$ be a grid in the half plane $\mathbb{R} \times \mathbb{R}_+$. Let

$$a = \max_{u \in M} |f'(u)| \quad \text{and} \quad b = \max_{u \in M} |g'(u)| \tag{3.1}$$

where the interval M depends on the initial data as well as on f, g and s, cf. Theorems 3.1 and 3.2 below. The scheme is derived by considering analytic solutions of parallel Riemann problems, originating from piecewise constant initial data. These parallel solutions will not interact if choosing the ratio of the mesh size of the grid

$$\lambda \equiv \frac{\tau}{\delta} < \frac{1}{2} \min\left(\frac{1}{a}, \frac{1}{b}\right). \tag{3.2}$$

It is assumed that the ratio is constant when $\tau, \delta \searrow 0$. Using U_i^j as the approximate solution at grid point (i, j) the scheme reads

$$U_i^{j+1} = U_i^j + \lambda\big(f(u_{i-1/2}^j) - f(u_{i+1/2}^j)\big), \quad i > 0 \tag{3.3}$$

$$U_0^{j+1} = U_0^j + \lambda\big(g(u_{-1/2}^j) - f(u_{1/2}^j) + S^j\big) \tag{3.4}$$

$$U_i^{j+1} = U_i^j + \lambda\big(g(u_{i-1/2}^j) - g(u_{i+1/2}^j)\big), \quad i < 0 \tag{3.5}$$

where

$$U_i^0 = \frac{1}{\delta} \int_{(i-1/2)\delta}^{(i+1/2)\delta} u_0(x)\, dx \qquad \text{and} \qquad S^j = \frac{1}{\tau} \int_{j\tau}^{(j+1)\tau} s(t)\, dt$$

and from the solution of the Riemann problem (1.6) it follows that the fluxes on the straight lines, in the t-direction, between the grid points are given by

$$(3.6)\quad h(u^j_{i-1/2}) = \begin{cases} \min\limits_{v\in[U^j_{i-1},U^j_i]} h(v), & \text{if } U^j_{i-1}\le U^j_i \\ \max\limits_{v\in[U^j_i,U^j_{i-1}]} h(v), & \text{if } U^j_{i-1}> U^j_i \end{cases} \quad \text{where} \quad h=\begin{cases} f, & i>0\\ g, & i\le 0.\end{cases}$$

Note that $h(u^j_{i-1/2}) = \hat{h}(U^j_{i-1},U^j_i) = \check{h}(U^j_i,U^j_{i-1})$. Define a piecewise constant function $\tilde{U}^\tau(x,t)$ by

$$\tilde{U}^\tau(x,t) = U^j_i \quad \text{for } (x,t)\in \big[(i-1/2)\delta,(i+1/2)\delta\big)\times\big[j\tau,(j+1)\tau\big).$$

The scheme (3.3)–(3.5) is conservative in the sense that the numerical solution preserves the same amount of mass as the analytical solution. A theorem of Lax and Wendroff [16], states that if a sequence of numerical solutions obtained by a conservative method applied to the equation $u_t + f(u)_x = 0$, $x\in\mathbb{R}$, is convergent, then it converges to a weak solution. This is also true for the scheme (3.3)–(3.5) and the proof is very similar to the one of the Lax-Wendroff Theorem.

3.2. Justification of Condition Γ. The justification is done in two ways. First we consider a time point when $u^\pm(t)$ are smooth and then when they are discontinuous.

Assume that the scheme (3.3)–(3.5) converges (in weak sense) to a solution of (1.2) with $u^\pm$ well defined. Let t_0 be a time such that $u^\pm(t_0) = u_\pm(t_0)$, and let $j\tau \le t_0 < (j+1)\tau$ hold as $j\to\infty$, $\tau\to 0$. Hence we assume that $U^j_{\pm i}\to u^\pm(t_0)$, $i=1,2$, $U^j_0\to u^0$ and $S^j\to s(t_0)$ as $j\to\infty$. This and the property $h(u^j_{i-1/2}) = \hat{h}(U^j_{i-1},U^j_i) = \check{h}(U^j_i,U^j_{i-1})$ for the cells -1, 0 and 1 in the scheme (3.3)–(3.5) yield

$$\begin{aligned} \hat{f}(u^0;u^+) &= \hat{f}(u^+;u^+)\\ \hat{f}(u^0;u^+) &= \check{g}(u^0;u^-)+s(t_0)\\ \check{g}(u^0;u^-) &= \check{g}(u^-;u^-). \end{aligned}$$

The second equality says that $u^0\in\bar{U}(t_0)$ and since $\hat{f}(u^+;u^+) = f(u^+)$ (and the same for g) holds we conclude that

$$f(u^+) = g(u^-)+s(t_0) = \hat{f}(u^0;u^+) \quad\Longleftrightarrow\quad (u^+,u^-)\in\Gamma,$$

i.e. Condition Γ is satisfied at all time points when $u^\pm$ are continuous.

Now to the case when $u^\pm(t)$ are discontinuous. For a general solution of (1.2), the "most common" cases concerning the intersection of the graphs of $\hat{f}$ and $\check{g}+s(t)$ are Cases 1 and 2 after Proposition 2.11. We shall apply the scheme (3.3)–(3.5) to the Riemann problem with discontinuous flux function (2.2), where $s\equiv 0$, in these two cases, see Theorems 3.1 and 3.2 below. Recall that the analytical solution, which satisfies Condition Γ, consists of two Riemann cones, one on each side of the t-axis. The boundary values on either side of the t-axis are constant for all $t\ge 0$, say $(u^+,u^-)\equiv(\alpha_0,\beta_0) = c(u_r,u_l,0)$. Because α_0 is constant, the solution of RP$(f;,\alpha_0,u_r)$ is, in $x>0$, $t>0$, identical to the solution of the quarter-plane problem

$$(3.7)\quad \begin{aligned} &u_t + f(u)_x = 0, && x>0,\ t>0\\ &u(x,0) = u_r, && x>0\\ &u(0,t) = \alpha_0, && t\ge 0. \end{aligned}$$

Hence the usual Godunov method produces this solution (in $x > 0$, $t > 0$) both when it is applied to $\mathrm{RP}(f; \alpha_0, u_r)$ and to (3.7). Notice that the constants u_r and α_0 imply that the limit of $\tilde{U}^\tau(x_0, t_0)$ when $\tau \searrow 0$ is obtained by considering the values on a fixed grid along a diagonal with speed x_0/t_0. In Theorems 3.1 and 3.2 below the sequences $\{U_1^j\}$ and $\{U_{-1}^j\}$ are shown to converge to the constants that satisfy Condition Γ, i.e. $(u^+, u^-) \equiv (\alpha_0, \beta_0) = c(u_r, u_l, 0)$. Thus the sequence $\{U_1^j\}$ lies "close" to the constant sequence $\{\alpha_0\}$, and by the reasoning above, using the latter sequence will produce a well defined entropy satisfying solution of (3.7).

THEOREM 3.1. *Assume that $f' > 0$, $g' < 0$ and that the graphs of $f \equiv \hat{f}$ and $g \equiv \breve{g}$ intersect at $\bar{u}$ as in Case 1 (after Proposition 2.11). Then the sequences U_{-1}^j and U_1^j converge to $\bar{u}$ as $j \to \infty$.*

Proof. In the definition of a and b, (3.1), let M be a connected and compact interval, which has $\bar{u}$ as the centre and contains u_l and u_r. We shall prove by induction that $U_i^j \in M$ for $i = -1, 0, 1$ so that the scheme (3.3)–(3.5) is well defined. This is by definition true for $j = 0$ because $U_0^0 = \frac{1}{2}(u_l + u_r)$ and $U_{-i}^0 = u_l$ and $U_i^0 = u_r$, $i = 1, 2, \ldots$ Assume that $U_i^j \in M$ for some $j \geq 0$, $i \in \mathbb{Z}$, then $f' > 0$, $g' < 0$ and (3.6) imply

$$\begin{aligned} g(u_{-i-1/2}^j) &= g(U_{-i}^j) \\ f(u_{i+1/2}^j) &= f(U_i^j) \end{aligned} \qquad i = 0, 1, 2, \ldots \tag{3.8}$$

Let $h = g - f$. Then $h(\bar{u}) = 0$ and $0 > h' = g' - f' \geq -b - a$ imply

$$-(a + b) \leq \frac{h(x)}{x - \bar{u}} < 0, \quad \forall x \in M \setminus \{\bar{u}\}. \tag{3.9}$$

Since $a, b > 0$, (3.2) implies

$$\lambda < \frac{1}{2} \min\left(\frac{1}{a}, \frac{1}{b}\right) = \frac{1}{2(a+b)} \min\left(\frac{a+b}{a}, \frac{a+b}{b}\right) \leq \frac{1}{a+b},$$

which together with (3.9) gives

$$0 < 1 + \lambda \frac{h(x)}{x - \bar{u}} < 1, \quad \forall x \in M \setminus \{\bar{u}\}. \tag{3.10}$$

Using (3.8) in the iteration formula (3.4) gives

$$U_0^{j+1} - \bar{u} = \left(1 + \lambda \frac{h(U_0^j)}{U_0^j - \bar{u}}\right) (U_0^j - \bar{u})$$

and hence (3.10) implies $U_0^{j+1} \in M$. For the 1-cell we have from (3.3)

$$U_1^{j+1} - U_0^j = (U_1^j - U_0^j) \left(1 - \lambda \frac{f(U_1^j) - f(U_0^j)}{U_1^j - U_0^j}\right). \tag{3.11}$$

Now the fact that $f' > 0$ together with the bound (3.2) implies that the last factor in (3.11) lies strictly between 1/2 and 1. This implies that U_1^{j+1} lies between U_1^j and U_0^j and thus $U_1^{j+1} \in M$. Repeating this procedure with the corresponding formula of

(3.11) for the 2-cell etc. we can obtain $U_i^{j+1} \in M$ for every $i = 1, 2, \ldots$ Analogously holds $U_{-i}^{j+1} \in M$ for every $i = 1, 2, \ldots$ and the induction is finished.

The iterative formulas (3.3) and (3.4) give the discrete system

$$\begin{cases} U_0^{j+1} = U_0^j + \lambda h(U_0^j) \\ U_1^{j+1} = U_1^j + \lambda\big(f(U_0^j) - f(U_1^j)\big) \end{cases} \tag{3.12}$$

The only fixed point for this system is $(\bar{u}, \bar{u})$ and the eigenvalues of the triangular functional matrix are $1 + \lambda h'(U_0^j)$, $1 - \lambda f'(U_1^j)$. By the bound (3.2) these eigenvalues have modulus < 1, hence $U_i^j \to \bar{u}$ as $j \to \infty$ for $i = 0, 1$. The corresponding procedure can be done with g instead of f to obtain $U_{-1}^j \to \bar{u}$ as $j \to \infty$. □

THEOREM 3.2. *Assume that $f' > 0$ and that g has precisely one stationary point, which is a global maximizer, see Figure* 2.6. *Let the intersection be as in Case* 2b *(after Proposition* 2.11*), with u_l and u_2 as in Figure* 2.6. *Assume that $u_l + u_r \leq 2u_2$. Then the sequences $U_{-1}^j \to u^- \equiv u_l$ and $U_1^j \to u^+ \equiv u_2$ as $j \to \infty$.*

The proof is found in [7]. The assumption $u_l + u_r \leq 2u_2$ is made to be able to apply an induction proof similar to that of Theorem 3.1. If u_r is larger we get, according to computer simulations, a transient behaviour before it is possible to apply the induction proof.

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