Introduction to the Scalar Non-Linear Conservation Law

Stefan Diehl
Department of Mathematics
Lund Institute of Technology
1996

Introduction

Many physical phenomena obey the *conservation law*: The change of the total amount of some physical entity (mass, momentum etc.) in a region of space is equal to the inward net flux across the boundary of that region (provided no sources or sinks are present). Such conservation laws are used to model phenomena in, for example, gas and fluid dynamics, traffic flow and sedimentation of solid particles in a liquid.

We shall study the conservation law in one dimension. As an example we shall consider traffic flow along a one way road. Take the x axis along the road and let $u(x,t)$ denote the density (number of cars per unit length) of the traffic at the point x at time t. Let f denote the flow rate of cars, i.e., the number of cars per unit time passing a point x at time t.

Let (x_1, x_2) be an arbitrary interval of the x axis. The conservation law written in mathematical terms in integral form is:

$$\underbrace{\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx}_{\text{increase of number of cars per unit time}} = \underbrace{f|_{x=x_1}}_{\text{flux in per unit time}} - \underbrace{f|_{x=x_2}}_{\text{flux out per unit time}} \tag{1}$$

If we assume that the concentration $u(x,t) \in C^1$, then the left-hand side of (1) can be written $\int_{x_1}^{x_2} \frac{\partial u}{\partial t}\,dx$. If the right hand side of (1) is written $-\int_{x_1}^{x_2} \frac{\partial f}{\partial x}\,dx$, then we get

$$\int_{x_1}^{x_2} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x}\right)\,dx = 0.$$

Since this holds for every interval (x_1, x_2) (and assuming that the integrand is continuous) it follows that

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad \Longleftrightarrow \quad u_t + f_x = 0. \tag{2}$$

This partial differential equation is called the *continuity equation* or *conservation law*.

A common dependence on x and t of the flux function f is of the form

$$f = f(u(x,t)), \tag{3}$$

where $f(u)$ is assumed to be a smooth function. For the traffic-flow problem it is natural to assume that the speed v of a car is dependent only on the local concentration of cars at the point x at time t, i.e., $v = v(u)$, and we can use the simple linear function $v(u) = v_0(1 - u/u_{\max})$, where v_0 is the free speed of a car (the limit speed of the road) and $u_{\max}$ is the maximal concentration of cars. The total traffic-flux function is thus the parabola

$$f(u) = v(u)u = v_0\left(1 - \frac{u}{u_{\max}}\right)u, \quad 0 \le u \le u_{\max}, \tag{4}$$

see Figure 1.

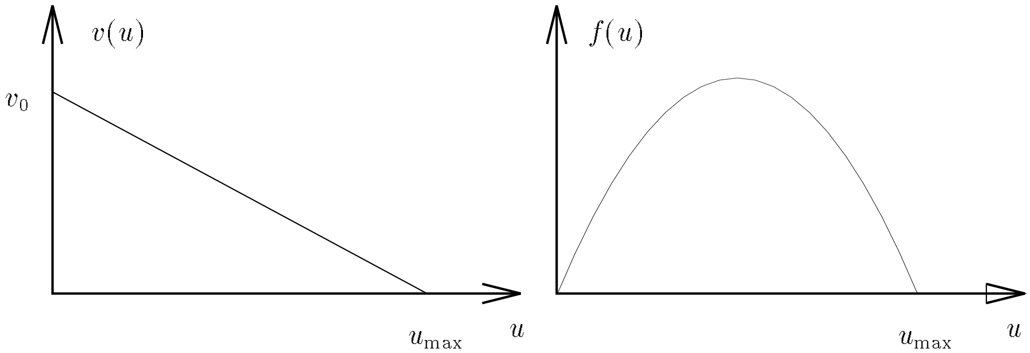

Figure 1: The car speed $v(u)$ and the traffic-flux function $f(u)$.

Using (3) we can write equation (2)

$$u_t + f(u)_x = 0 \iff u_t + f'(u)u_x = 0. \tag{5}$$

This equation is called *quasi-linear*, since it is linear in the derivatives but the coefficient of u_x depends on u.

Characteristics

To be able to solve the partial differential equation (5) some initial concentration distribution must be given at $t = 0$. Then we get the initial value problem

$$\begin{aligned} &u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{6}$$

Let $x = x(t)$ be a level curve in the x-t plane, i.e.,

$$u(x(t), t) = \text{ constant } = U_0.$$

Differentiating with respect to t and using $u_t = -f'(u)u_x$ gives

$$0 = u_x x'(t) + u_t = u_x(x'(t) - f'(U_0)).$$

Generally, it must hold that $x'(t) = f'(U_0)$, which means that the level curve is a straight line with slope $= 1/f'(U_0)$ in the x-t plane. Such a line is called a *characteristic*. (If $u_x \equiv 0$ in some region, then the conservation law implies that $u_t \equiv 0$ too, hence the solution is constant, say $\equiv U_0$, and we can still define the level curves to be straight lines by the equation $x'(t) = f'(U_0)$). The value $f'(U_0)$ is called the *signal speed*, since a wavefront or a disturbance will propagate with this speed. Note the non-linearity consisting in that the signal speed is dependent on the solution u. A geometrical construction of a solution for given initial data $u(x,0) = u_0(x)$ can be done as follows: Through each point x_0 on the x axis, draw a straight line with speed $f'(u_0(x_0))$ in the x-t plane. Along this line the solution has the value $u_0(x_0)$. Analytically we can write the solution in implicit form: The connection between the value u, the point x and time t is

$$\begin{cases} x = f'(u_0(x_0))t + x_0 \\ u = u_0(x_0). \end{cases} \tag{7}$$

Example 1. Consider the initial value problem

$$u_t + uu_x = 0$$
$$u_0(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1. \end{cases}$$

Here $f(u) = \frac{1}{2}u^2$ and the equation is called the inviscid Burgers' equation. Note that the signal speed is simply $f'(u) = u$, so it is easy to construct a solution geometrically, see Figure 2.

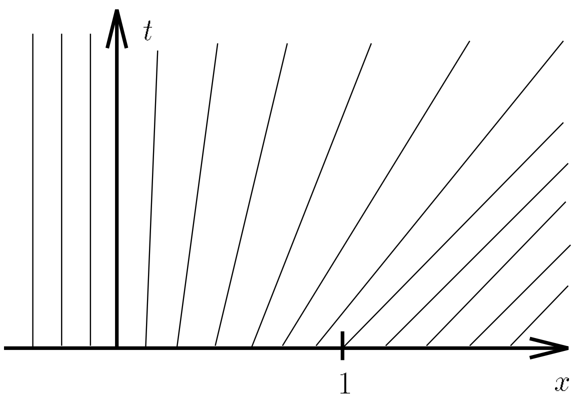

Figure 2: Geometric solution of Example 1.

The analytic formulae (7) give, for example, for the characteristics starting at $0 \leq x \leq 1$, where $u_0(x) = x$:

$$\begin{cases} x = x_0 t + x_0 \\ u = x_0 \end{cases} \quad \Longrightarrow \quad u(x,t) = \frac{x}{1+t},$$

and the complete solution can be written

$$u(x,t) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{1+t}, & 0 \leq x \leq 1+t \\ 1, & x \geq 1+t \end{cases} \qquad t > 0.$$

□

Example 2. Consider the initial value problem

$$u_t + uu_x = 0$$

$$u_0(x) = \begin{cases} 1, & x \le 0 \\ 1 - x, & 0 \le x \le 1 \\ 0, & x \ge 1. \end{cases}$$

Drawing the characteristics starting from the x axis we get the picture of Figure 3.

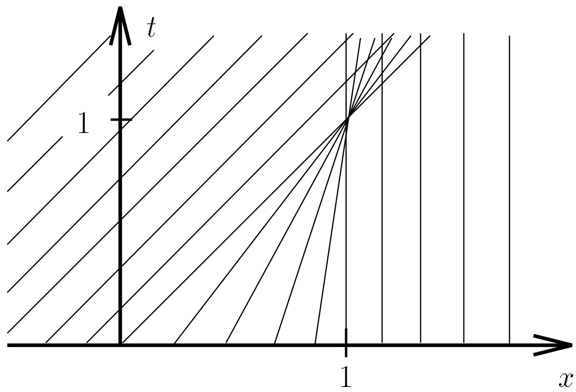

Figure 3: Geometric solution of Example 2.

We see that a continuous solution can not be defined beyond $t = 1$, because characteristics with different values intersect; a discontinuity appears. □

Shock Waves and the Jump Condition

In Example 2 we saw that a continuous solution could not be defined after a time when characteristics intersect. Even for differentiable initial data, $u_0(x) \in C^1$, discontinuous solutions may appear after a finite time. This can be seen in the following way. A C^1 solution is obtained if we can solve for x_0 in the first equation of (7), thus formally $x_0 = x_0(x,t)$, and then substitute this expression into the second equation of (7). According to the implicit function theorem this can be done if

$$\frac{dx}{dx_0} = f''(u(x_0))\, u_0'(x_0)t + 1 \neq 0. \tag{8}$$

This is true for small $t > 0$, which proves that if the function $u_0(x)$ is smooth, then there exists a smooth solution $u(x,t)$ for small $t > 0$. The smallest time for which $\frac{dx}{dx_0} = 0$ holds is called the *critical time.*

To be able to continue the solution after a discontinuity appears, one has to generalize the concept of solution. The conservation law in (5) is multiplied by a test function φ and after partial integration one arrives at the condition

$$\int_0^{\infty}\int_{-\infty}^{\infty} (u\varphi_t + f(u)\varphi_x)\, dx\, dt + \int_{-\infty}^{\infty} u(x,0)\varphi(x,0)\, dx = 0, \quad \forall \varphi \in C_0^1 \tag{9}$$

where C_0^1 stands for continuously differentiable functions with compact support. A function u that satisfies (9) is called *weak* solution of the conservation law (5).

The conservation law also states how a discontinuity moves. Let u be a piecewise C^1 solution of the conservation law, with a discontinuity curve $x = x(t) \in C^1$ in the x-t plane, and let (a, b) be an interval parallel to the x axis in such a way that the curve $x(t)$ intersects the interval at a time t, see Figure 4. Let $u^{\pm} = u(x(t) \pm 0, t)$ denote the values

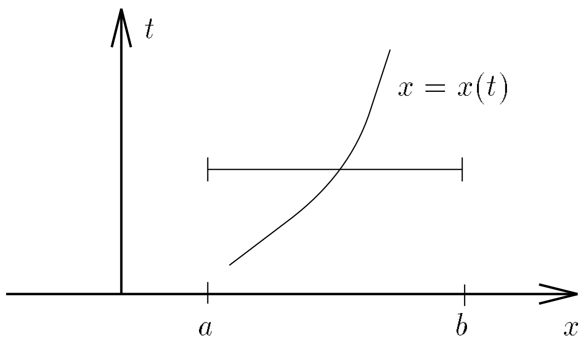

Figure 4: A discontinuity curve $x = x(t)$.

of the solution to the left and right of the discontinuity curve. The conservation law (1) for the interval (a, b) gives

$$
\begin{aligned}
f(u(a,t)) - f(u(b,t)) &= \frac{d}{dt}\int_a^b u\,dx = \frac{d}{dt}\Big(\int_a^{x(t)} u\,dx + \int_{x(t)}^b u\,dx\Big) = \\
&= \int_a^{x(t)} u_t\,dx + u^- x'(t) + \int_{x(t)}^b u_t\,dx - u^+ x'(t) = [u_t = -f_x] = \\
&= f(u(a,t)) - f(u(b,t)) + f(u^+) - f(u^-) - (u^+ - u^-)x'(t)
\end{aligned}
$$

which means that the speed of the discontinuity satisfies

$$
\text{(RH)} \qquad x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-},
$$

the so called *jump condition* or *Rankine-Hugoniot condition.* Note that the speed is the slope of the straight line through the points $(u^-, f(u^-))$ and $(u^+, f(u^+))$ on the graph of f.

It can be shown that if $u(x, t)$ is a piecewise smooth function, which satisfies the initial data $u(x, 0) = u_0(x)$, then $u(x, t)$ is a weak solution of (9) if and only if

- the conservation law is satisfied at points where $u \in C^1$,
- the jump condition (RH) is satisfied at discontinuities.

Example 2 (continued). The jump condition (RH) gives that the discontinuity starting at $(x, t) = (1, 1)$ has the speed $(f(u) = \frac{1}{2}u^2)$

$$
x'(t) = \frac{f(1) - f(0)}{1 - 0} = \frac{1}{2}.
$$

□

Viscous Waves and the Entropy Condition

The problem by introducing generalized solutions (weak solutions) is that we may obtain different solutions for the same initial data as the following example shows.

Example 3. The geometric solution of the problem

$$\begin{aligned} u_t + uu_x &= 0 \\ u_0(x) &= \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \end{aligned}$$

is shown in Figure 5.

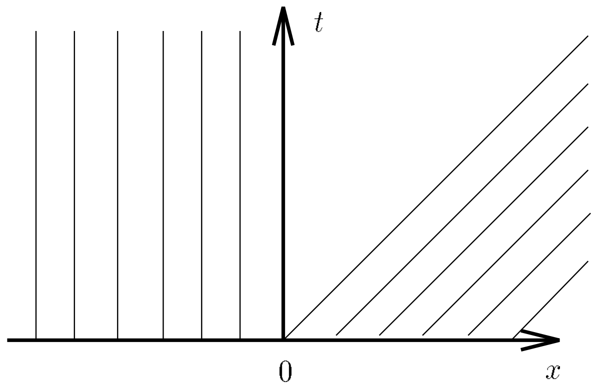

Figure 5: Geometric solution of Example 3.

To fill the gap $0 < x < t$ we can use the discontinuous solution

$$u(x,t) = \begin{cases} 0, & x \leq \frac{t}{2} \\ 1, & x > \frac{t}{2} \end{cases} \quad t > 0, \tag{10}$$

which satisfies (RH), or the solution with an *expansion wave*

$$u(x,t) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{t}, & 0 < x \leq t \\ 1, & x > t \end{cases} \quad t > 0, \tag{11}$$

which is a continuous solution. □

In order to select a unique, physically relevant solution, an additional condition must be imposed. We shall now motivate the so called *entropy condition*, which will pick out physically correct shocks and discard others. This condition can be obtained by studying what happens when diffusion or viscosity is also taken into account. Use $f := f(u) - \varepsilon u_x$ in the derivation instead of (3). The term $-\varepsilon u_x$, with $\varepsilon > 0$, comes from Fick's law. Then we obtain the viscous equation

$$u_t + f(u)_x = \varepsilon u_{xx}. \tag{12}$$

Generally, solutions of (12) are smooth for $\varepsilon > 0$. If ε is small we get approximately the same solutions as of the conservation law (5), but the shocks are now smoothed, see Figure 6.

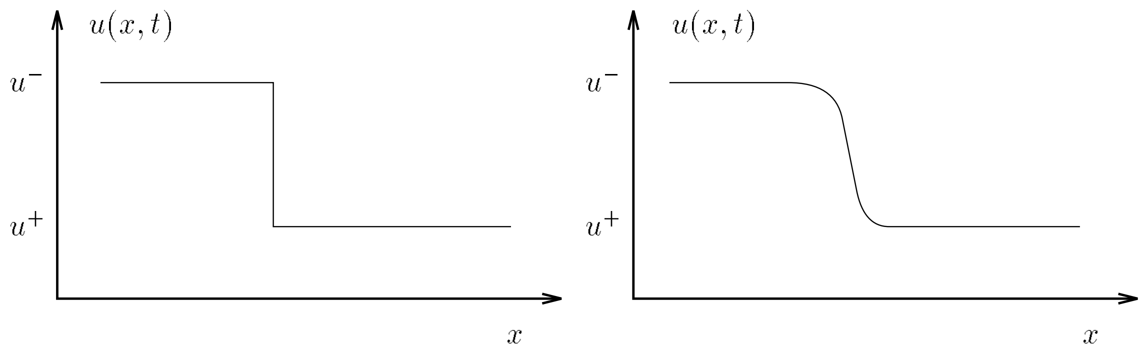

Figure 6: A wave solution of equation (12) when $\varepsilon = 0$ (left) and $\varepsilon > 0$ (right).

Consider a solution of equation (5) consisting of a single discontinuity with speed s and with the left and right limits u^- and u^+. We say that such a shock is allowed if it satisfies the *viscous profile condition*: For given constants u^+, u^- and

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-},$$

according to the jump condition (RH), there exists a travelling-wave solution, or viscous profile,

$$u(x,t) = v(\xi) \quad \text{with} \quad \xi = \frac{x - st}{\varepsilon} \tag{13}$$

of the equation (12) with $v(\xi) \to u^\pm$ as $\xi \to \pm\infty$. Hence $\varepsilon \searrow 0$ implies that the travelling wave converges to the wanted shock wave with speed s. Substituting (13) into (12) yields $v'' = f(v)_\xi - sv'$, hence

$$v' = f(v) - sv + C \tag{14}$$

for some constant C. Letting $\xi \to \pm\infty$ and using the jump condition, it follows that $C = -f(u^-) + su^- = -f(u^+) + su^+$ and that both u^+ and u^- are fixed points of (14). Hence the viscous profile $v(\xi)$ satisfies the ordinary differential equation

$$v' = f(v) - f(u^-) - s(v - u^-) \equiv \psi(v). \tag{15}$$

Assume that $u^- > u^+$ as in Figure 6. If $v'(\xi_0) = \psi(v(\xi_0)) = 0$ for some ξ_0, then $v(\xi) \equiv v(\xi_0)$ is the unique solution of (15). Because $v(\xi) \to u^\pm$ as $\xi \to \pm\infty$, v is either strictly increasing or decreasing. The only possibility is thus that $v'(\xi) = \psi(v(\xi)) < 0$ for all $\xi \in \mathbb{R}$. Hence $\psi(v) < 0$ for $v \in (u^+, u^-)$, or equivalently

$$s < \frac{f(v) - f(u^-)}{v - u^-} \quad \text{for all } v \text{ strictly between } u^- \text{ and } u^+, \tag{16}$$

which says that the graph of f should lie strictly below the straight line from $(u^-, f(u^-))$ to $(u^+, f(u^+))$. Hence the shock in Example 2 is allowed. If $u^- < u^+$, (16) still holds and says that the graph of f lies above the straight line from $(u^-, f(u^-))$ and $(u^+, f(u^+))$.

The solution (10) in Example 3 has a shock which is not allowed. The correct physical solution is (11).

Condition (16) is thus a necessary condition for an admissible shock. The entropy condition by Oleinik

$$s \leq \frac{f(v) - f(u^-)}{v - u^-} \quad \text{for all } v \text{ between } u^- \text{ and } u^+ \tag{17}$$

is a sufficient condition for uniqueness of a weak solution of the initial-value problem (6) for general functions $f(u)$. Letting $v \to u^-$ in (16) or (17) yields $f'(u^-) \geq s$. Analogously, $f'(u^+) \leq s$ holds. For a strictly convex or concave $f(u)$, as in our examples above, these inequalities are strict, and the entropy condition becomes

$$f'(u^-) > x'(t) > f'(u^+), \tag{18}$$

i.e., *characteristics on either side of the shock, when continued in the direction of increasing t, intersect the shock.*

Exercises

1. Show that both $u = x/(1+t)$ (cf. Example 1) and $u = x/t$ (cf. Example 3) satisfy Burgers' equation.

2. Write explicitly the solution of Example 2 and draw the graphs of $u(x, 1/2)$, $u(x, 1)$ and $u(x, 3/2)$.

3. In the gap of Figure 5, draw the characteristics of the two solutions (10) and (11).

4. Write (7) as one equation without x_0. This equation defines u implicitly for small $t > 0$ if the initial data $u_0(x)$ is smooth. Give explicit formulae for u_t and u_x. Check that $u_t + f'(u)u_x = 0$. What happens to u_t and u_x when $t \to$ a critical time?

5. Solve the following problem

$$u_t + uu_x = 0$$
$$u_0(x) = \begin{cases} 2, & x \leq 0 \\ 0, & 0 < x \leq 1 \\ -1, & x > 1. \end{cases}$$

6. Solve the traffic-flow problem with the initial data

$$u_0(x) = \begin{cases} 30, & x \leq 0 \\ 30 + 60x, & 0 < x \leq 1 \\ 90, & x > 1. \end{cases}$$

Let $v_0 = 100$ km/h and $u_{\max} = 100$ cars/km.

7. What shocks are admitted according to the entropy condition in the case of the traffic-flux function (4)? Note how the entropy condition rejects unrealistic shocks.

8. Show the inequality $s > f'(u^+)$ in (18) by using $C = -f(u^+) + su^+$ in the derivation.

9. The modelling of a queue of cars that starts driving when the red light (at $x = 0$) turns into green can be done by solving (5) with the initial data

$$u_0(x) = \begin{cases} 100, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

($v_0 = 100$ km/h and $u_{\max} = 100$ cars/km). Solve this problem.
Hint: Fill the gap with a fan of characteristic lines (an expansion wave).

Answers

2.

$$u(x,t) = \begin{cases} 1, & x \le t \\ \frac{1-x}{1-t}, & t < x \le 1 \\ 0, & x > 1 \end{cases} \qquad 0 < t < 1$$

$$u(x,t) = \begin{cases} 1, & x \le \frac{1}{2}(t+1) \\ 0, & x > \frac{1}{2}(t+1) \end{cases} \qquad t \ge 1.$$

4. $u = u_0(x - f'(u)t), \quad u_t = -\dfrac{u_0' f'(u)}{u_0' f''(u)t + 1}, \quad u_x = \dfrac{u_0'}{u_0' f''(u)t + 1}.$

5. Two shocks emanating from $(x,t) = (0,0)$ and $(1,0)$ will meet at the point $(2/3, 2/3)$, and beyond this point there is a single shock with speed $1/2$.

6. A shock will develop after 30 seconds at $x = 1/3$ km. The shock speed is -20 km/h.

7. A shock must satisfy $u^- < u^+$, see also Exercise 9.

9.

$$u(x,t) = \begin{cases} 100, & x \le -100t \\ 50 - \frac{x}{2t}, & -100t < x \le 100t \\ 0, & x > 100t \end{cases} \qquad t > 0.$$