Under the assumption that image points are corrupted by Gaussian noise, minimize the reprojection error.

The reprojection error

In regular coordinates \((x = (x, y))\) the projection is

\[
\begin{pmatrix}
\frac{p^1_x}{p^3_x} & \frac{p^2_x}{p^3_x}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

\(P^1, P^2, P^3\) are the rows of \(P\).

The reprojection error is

\[
\left\| \left( x - \frac{p^1_x}{p^3_x}, y - \frac{p^2_x}{p^3_x} \right) \right\|^2.
\]
Framework: Affine Projective Estimation

\[
    r_i(x) = \frac{(a_i^T x + \tilde{a}_i)^2 + (b_i^T x + \tilde{b}_i)^2}{(c_i^T x + \tilde{c}_i)^2}, \quad c_i^T x + \tilde{c}_i > 0.
\]

Solve either the projective least-squares problem

\[
    \min_{\{x; c_i^T x + \tilde{c}_i > 0, \ \forall i\}} \sum_i r_i(x),
\]

or the min-max problem (easier)

\[
    \min_{\{x; c_i^T x + \tilde{c}_i > 0, \ \forall i\}} \max_i r_i(x).
\]
Quotients of Affine Functions

If

\[ P = \begin{bmatrix} -A^1 & t_1 \\ -A^2 & t_2 \\ -A^3 & t_3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X \\ 1 \end{bmatrix}. \]

Then

\[
\left\| \left( x - \frac{P^1 X}{P^3 X}, y - \frac{P^2 X}{P^3 X} \right) \right\|^2 =
\left\| \left( \frac{x_1(A^3 X + t_3) - (A^1 X + t_1)}{A^3 X + t_3}, \frac{x_2(A^3 X + t_3) - (A^2 X + t_2)}{A^3 X + t_3} \right) \right\|^2.
\]

Least squares problem with quotients of affine functions. If either \(A\) or \(X\) is known!
Framework Examples: Triangulation

Given
- Image data (2D points - \(x\))
- Cameras (\(P\))

Estimate
- Structure (3D points - \(X\))

\[
\begin{bmatrix}
\frac{x_1(A^3X+t_3)-(A^1X+t_1)}{A^3X+t_3} \\
\frac{x_2(A^3X+t_3)-(A^2X+t_2)}{A^3X+t_3}
\end{bmatrix}
\|^2
\]

Quotients of affine functions in \(X!\)
Framework Examples: Resection (uncalibrated)

**Given**
- Image data (2D points - $x$)
- Structure (3D points - $X$)

**Estimate**
- Cameras ($P$)

Quotients of affine functions in $A, t$!
Framework Examples: SfM with Known Orientations

**Given**
- Image data (2D points - \( u \))
- Camera orientations (\( A \))

**Estimate**
- Structure (\( U \)), Positions (\( t \)).

\[
\begin{align*}
\| & \left( \frac{x_1(A^3X+t_3) - (A^1X+t_1)}{A^3X+t_3} \right) \\
& \left( \frac{x_2(A^3X+t_3) - (A^2X+t_2)}{A^3X+t_3} \right) \|_2
\end{align*}
\]

Quotients of affine functions in \( X \) and \( t \)!
Framework: Affine Projective Estimation

Why use the projective least-squares formulation?

\[
\min_{\{x; c_i^T x + \tilde{c}_i > 0, \forall i\}} \sum_i r_i(x),
\]

- Geometrically meaningful goal function (minimize reprojection error).
- Statistically optimal (under the assumption of Gaussian noise).

Why the min-max problem?

\[
\min_{\{x; c_i^T x + \tilde{c}_i > 0, \forall i\}} \max_i r_i(x).
\]

- Geometrically meaningful goal function (minimize reprojection error).
- Easier to minimize due to convexity properties.
Global Optimization

See lecture notes.
Global Optimization

Checking if there is

\[ \mathbf{X} \in \bigcap_{i \in I} \{ \mathbf{X}; \quad r_i(\mathbf{X}) \leq \epsilon^2, \quad P_i^3 \geq \delta \} \]

is a convex problem:

\[
\begin{align*}
\min_{s, \mathbf{X}} & \quad s \\
\text{s.t.} & \quad \left\| \left( (x_i P_i^3 - P_i^1) \mathbf{X}, (y_i P_i^3 - P_i^2) \mathbf{X} \right) \right\| \leq \epsilon P_i^3 \mathbf{X} + s, \quad \forall i \in I \\
& \quad P_i \mathbf{X} \geq \delta, \quad \forall i \in I.
\end{align*}
\]

If \( s > 0 \) then the set is empty!
Global Optimization: Triangulation
Global Optimization: Triangulation
Global Optimization: Triangulation

The 3D point must lie in the intersection of the cones.
Reduce the size of the cones $\Leftrightarrow$ lower the permitted error.
As long as there is a point in the intersection.
No point in the intersection.
Global Optimization: Triangulation

**Algorithm**

Minimizes the maximal reprojection error. Finds the smallest possible $\epsilon$ for which there is a solution $X$ with all reprojection errors is less than $\epsilon$. That is, solves

$$\min_X \max_i r_i(X).$$

1. Let $\epsilon_l$ and $\epsilon_u$ be lower and upper bound on the optimal error.
2. Check if there is a solution such that

$$r_i(X) \leq \frac{\epsilon_u + \epsilon_l}{2}, \quad \forall i$$

(convex optimization problem).
3. If there is set $\epsilon_u = \frac{\epsilon_u + \epsilon_l}{2}$, otherwise set $\epsilon_l = \frac{\epsilon_u + \epsilon_l}{2}$.
4. If $\epsilon_u - \epsilon_l > tol$ (some predefined tolerance) goto 2.
Global Optimization

Generalizations

Works for other problems as well:

- Computing camera matrix given 3D-points and projections.
  \[
  \begin{bmatrix} x_i \end{bmatrix}_\text{known} \sim P \begin{bmatrix} X_i \end{bmatrix}_\text{unknown known}
  \]

- Homography estimation.
  \[
  \begin{bmatrix} y_i \end{bmatrix}_\text{known} \sim H \begin{bmatrix} x_i \end{bmatrix}_\text{unknown known}
  \]

- Structure and motion if camera orientations are known.
  \[
  \begin{bmatrix} x_{ij} \end{bmatrix}_\text{known} \sim \begin{bmatrix} R_i \end{bmatrix}_\text{known} \begin{bmatrix} t_i \end{bmatrix}_\text{unknown} \begin{bmatrix} X_j \end{bmatrix}_\text{unknown}
  \]