Content: Travelling waves

10.5-10.10

Introduction

Travelling waves are usually associated with the one-dimensional wave equation

\[ c^2 u_{tt} = u_{xx} \]

for which the general solution is given by

\[ u(x,t) = f(x - ct) + g(x + ct). \]

The functions \( f \) and \( g \) are determined by the Cauchy-data, i.e., the initial conditions. For the diffusion equation

\[ u_t = u_{xx}, \]

no such travelling waves exist; if \( u(x,0) = \phi(x) \), the solution is given by

\[ u(x,t) = \phi'(K_t \ast \phi)(x), \quad K_t(x) = \frac{1}{\sqrt{4 \pi t}} e^{-x^2/4t}. \]

However, it turns out that a “slight” modification of it can produce travelling waves:

\[ u_t = u_{xx} + f(u), \]

where \( f \) is nonlinear, such waves do quite often exist. We first discuss the simplest such case where

\[ f(u) = u(1-u). \]

Consider the initial function below (black):

The solution to the problem is shown for three different times in blue. What we see is that on each axis \( u(x,t) \to 1 \) as \( t \to \infty \), but also that these functions take on a very specific form: there is a function \( \xi \) such that

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But what would the solution look like if we start out with an initial function which takes on values both below and above \( a \)?

We should have that \( u(.,t) \to 1 \) or \( u(.,t) \to 0 \) as \( t \to \infty \). But which?

It actually depends on which of the states \( u = 1 \) and \( u = 0 \) dominates, as measured by the sign of

\[ \int_0^1 f(u) \, du. \]

In fact, what is true is that there is a unique \( c \) such that the problem

\[ U'' + cU' + f(U) = 0, \quad U(-\infty) = 1, \quad U(\infty) = 0, \]

has a unique solution \( U_c \). Here \( c \) has the same sign as the integral above and

\[ u(x,t) \to U_c(x-ct), \quad x > 0 \text{ as } t \to \infty, \]

\[ u(x,t) \to U_c(-x-ct), \quad x < 0 \text{ as } t \to \infty, \]

at least if \( \phi > a \) on a sufficiently large interval. Hence we have two cases: if \( c > 0 \) we have a pair of diverging fronts and \( u(.,t) \to 1 \) as \( t \to \infty \), and if \( c < 0 \) they converge and \( u(.,t) \to 0 \) as \( t \to \infty \).

Existence of travelling fronts

If \( U(z) \) is a function such that \( u(x,t) = U(x-ct) \) is a solution of the PDE

\[ u_t = u_{xx} + f(u), \]

it must satisfy the second order equation

\[ U'' + cU' + f(U) = 0. \]

We are only interested in those travelling waves for which \( U(z) \) approaches distinct limits as \( z \to \pm \infty \), and these must clearly be zeros of \( f \). It is no restriction to assume them to be 0 and 1 and we therefore impose the boundary conditions

\[ U(-\infty) = 1, \quad U(\infty) = 0, \quad 0 \leq U(z) \leq 1. \]

We will assume that \( f(0) = f(1) = 0 \). Furthermore, we normalize \( U \) so that

\[ U(0) = 1/2 \]

in order to be able to talk about the travelling front.

The second order equation is equivalent to the first order system

\[ u' = p, \quad p' = -cp - f(u) \]

and the vector field at the \( u \)-axes points downward, telling us that any trajectory joining the two critical points \((1,0)\) and \((0,0)\) must satisfy \( p < 0 \) everywhere. Thus we have that

\[ U'(z) < 0 \text{ for all } z. \]

Because of this the curve \( z \to (U(z),U'(z)) \) can be represented by a graph

\[ p = P(u), \quad 0 \leq u \leq 1, \]

with \( P(u) < 0 \) for \( 0 < u < 1 \). This function \( P(u) \) satisfies the equation

\[ P'(u) + \frac{f(u)}{P(u)} = -c, \quad P(0) = P(1) = 0. \]
Obviously the converse is also true. Once we have found $P(u)$ we can solve the equation

$$U'(z) = P(U(z)), \quad U(0) = 1/2.$$ 

Note that

$$\int_0^{U(z)} \frac{du}{P(u)} = z,$$

so we see that we get a monotone function defined for all $z$ iff the integral of $du/P(u)$ is divergent both in $u = 0$ and $u = 1$. It can be shown that this is the case. Also note that

$$c \int_0^1 f(u) du = -\int_0^1 f(u) du,$$

and since $P < 0$ we thus get that $c$ has the same sign as $\int_0^1 f(u) du$.

We now come to the question of existence of travelling waves. As we have seen, these are trajectories in $p < 0$ in the phase-space connecting $(1,0)$ and $(0,0)$, so some information about their existence can be deduced from studying these singular points. Let the singular point be $(a,0)$. Then the linearisation around $(a,0)$ has the matrix

$$\begin{pmatrix} 0 & 1 \\ -f'(a) & -c \end{pmatrix},$$

which has eigenvalues

$$\lambda_\pm = \lambda_\pm (c) = \frac{1}{2} (-c \pm \sqrt{c^2 - 4f'(a)})$$

with corresponding eigenvectors

$$e_\pm = (1, \lambda_\pm).$$

If $c^2 < 4f'(a)$, the eigenvalues are complex, which means that the trajectories to/from $(a,0)$ spiral around the point and thus cannot stay in $p < 0$. There are two more cases. If $f'(a) > 0$ and $c^2 > 4f'(a)$ both eigenvalues have the same sign (depending on the sign of $c$) so either all trajectories leave $(a,0)$ (if $c < 0$) or enter $(a,0)$ (if $c > 0$) without a spiral behaviour. If $f'(a) < 0$ both eigenvalues are real but of opposite sign, so we have a saddle point in this case.

Ignoring the border line cases $f'(a) = 0$ we therefore have three possibilities:

(I) $(1,0)$ and $(0,0)$ are saddle points, i.e.

$$f'(0) < 0, f'(1) < 0;$$

(II) $(1,0)$ is a saddle point and $(0,0)$ a stable node, i.e.

$$f'(0) > 0, f'(1) < 0$$

in which case we must have $c \geq 2\sqrt{f'(0)}$;

(III) $(1,0)$ is an unstable node and $(0,0)$ a saddle point, i.e.

$$f'(0) < 0, f'(1) > 0.$$

But (III) is easily reduced to (II) by the simple transformation

$$\bar{u} = 1 - u, \quad \bar{f}(u) = -f(1 - u),$$

so we only have to consider (I) and (II). From now on we therefore assume $u = 1$ is a saddle point and we need the outgoing trajectory from $(1,0)$ to enter $(0,0)$ without leaving the fourth quadrant.

In case (I) this means that this trajectory must be the same as the single ingoing trajectory to $(0,0)$. Varying $c$ it can be shown that there is a unique $c$ for which this happens and we know that $c$ then must have the same sign as $\int_0^1 f(u) du$.

In case (II) it should be simpler. Here we need the outgoing trajectory from $(1,0)$ to be “caught” by one orbit that goes into $(0,0)$. Since all orbits close enough to the origin does so, this should not be too hard. It can be shown that this happens for all $c$ above a lower limit.

Remark Consider a region

$$D = \{(u,p); 0 < u < 1, -p(u) < p < 0, p(0) = \rho(1) = 0\}.$$

If we can find a function $\rho$ such that the vector $(u',p')$ points into the region everywhere, there must be such a trajectory. Such a region must contain the unstable trajectory from $(1,0)$ (exercise). On $p = 0$ we have $(u',p') = (0,-f(u))$ which obviously points into $D$. A tangent vector to the lower boundary is given by $(1,-\rho'(u))$, so $(\rho'(u),1)$ is an inward pointing normal to it, and the vector $(u',p')$ points into the region iff

$$0 \leq (\rho'(u),1) \cdot (u',p') = \rho'(u)p - cp - f(u) = -\rho'(u)p(u) + cp(u) - f(u),$$

which is equivalent to

$$c \geq \sup_{[0,1]} \{\rho'(u) + \frac{f(u)}{\rho(u)}\}.$$

Example For $f(u) = u(1-u)$ we have $f'(0) = 1$ and $f'(1) = -1$, so we must have $c \geq 2$ for there to exist a trajectory. They do all in fact exist, but in computer simulations only the case $c = 2$ occurs. The reason is that the speed of the wave depends on the shape of $u_0$.

Example For $f(u) = u(1-u)(u-a)$, $a \in (0,1)$, we have $f'(0) = -a < 0$ and $f'(1) = a - 1 < 0$, so there is at most one unique travelling wave solution. For future reference we can note that in this case

$$\int_0^1 f(u) du = \frac{1}{2a},$$

which is positive iff $a < 1/2$. In that case there is an explicit solution: the “Huxley pulse”

$$U(z) = \frac{1}{\exp(z/2) + 1},$$

which has speed $c = \frac{1}{\sqrt{2}} - a \sqrt{2}$. Note that for $a = 1/2$ this front is stationary.

Example In the budworm model with dispersion we have $Q$ is a fixed number

$$u_t = u_{xx} + f(u; R), \quad f(u; R) = u - u^2/2 + u\sqrt{R(1 + u^2)}.$$

When $R_1(Q) < R < R_2(Q)$ we have four zeros of $f(u; R)$: 0 (no budworms), $u_1(R)$ (low endemic state), $u_2(R)$ (unstable) and $u_3(R)$ (outbreak state).

For a forest which is in low endemic state, there is a localized outbreak. Will it spread? If the outbreak is large enough, we know that $u(x,t)$ will approach two travelling waves going in opposite directions, with a speed which has the same sign as

$$C(R) = \int_{u_1(R)}^{u_3(R)} f(u; R) du.$$

It can be shown (see exercises) that this function is strictly increasing with a unique zero in a point $R_0(Q)$. Hence, if $R < R_0(Q)$, the outbreak will die out by itself, whereas if $R > R_0(Q)$ it will spread over the whole forest. One way to stop the outbreak is to defoliate the whole forest (decreasing $R$ below the threshold), though it is not an appealing method.

It can be shown that $R(Q) > R_2(Q)$, and we know that for a spatially homogeneous forest the critical value for outbreak is $R_2(Q)$. Allowing for diffusion in the model increases the critical foliage density for an outbreak!
**Travelling waves in the plane**

In two (space) dimensions the corresponding equation would be

\[ u_t = \Delta u + f(u), \]

and a travelling wave starting in the origin would have the form \( u(x, t) = U(r - ct) \), where \( r = \sqrt{x_1^2 + x_2^2} \). However, for a radial function \( u(x, y, t) = u(r, t) \) the original equation would be

\[ u_t = u_r + \frac{1}{r} u_r + f(u), \]

so the equation for \( U \) should be

\[ -cu'' = U'' + \frac{1}{r} U' + f(U), \]

which does not work because of the middle term on the right hand side. But this disappear as \( r \to \infty \), so asymptotically we could have something that looks a little like travelling waves also in the plane. The following graph shows the spread of the Black Death 1347-1352 in Europe.

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**Example** A simple model of the spread of an epidemic which might be particularly relevant to the spread of rabies by foxes is

\[
\begin{align*}
I_t &= D I I + K I S - \mu I, \\
S_t &= -K I S,
\end{align*}
\]

where both \( I \) and \( S \) are population densities. The diffusion term is motivated by the fact that infected foxes often lose their territorial behaviour and invade ranges of neighbouring susceptibles. We assume an initial uniform spread \( S_0 \) of susceptibles and nondimensionalize:

\[ u = I/S_0, \quad v = S/S_0, \quad x' = \sqrt{K S_0/D} x, \quad t' = K S_0 t, \quad r = \mu/K S_0. \]

This gives us the model (drop asterix)

\[ u_t = \Delta u + u v - r v, \quad v_t = -u v, \quad u(x,0) \geq 0, \quad v(x,0) = 0. \]

We want to see if it has a one-dimensional travelling wave solution \( u(x, t) = f(x - ct), \quad v(x, t) = g(x - ct) \), which solves

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d^2 f}{dx^2} + c f + f g - r f = 0, \\
\frac{d g}{dx} = f g/c, \quad g(\infty) = 1
\end{array} \right.
\]

First we note that \( g \) is increasing, so that \( a = g(-\infty) \) exists and \( 0 < a < 1 \).

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We can now prove that there exist travelling wave solutions iff \( r < 1 \). In such a case there is a unique wave for every \( c \geq c_0 = 2\sqrt{1-r} \) and we have that \( a \) is the solution of the equation \( a - r \ln a = 1 \). Similar to Fishers equation. Also, with localized outbreaks, the wave will travel with the lower speed \( c_0 \).

**Exercises**

**Exercise 1** This exercise discusses steady state solutions to the equation

\[ u_t = u_{xx} + f(u), \]

i.e. solutions such that \( u(x, t) = \varphi(x) \).

1. Show that a steady state solution \( \varphi(x) \) solves the equation

\[ \frac{c^2}{2} + F(\varphi) = E, \quad F(\varphi) = \int_0^\varphi f(t) dt, \]

for some \( E \).

2. Use this to show that the range of the solution can be obtained from the graph of \( F(\varphi) \). Also show that if it is bounded, it has a single maximum and is symmetric around the value in which this is attained.

3. If we normalize a bounded solution so that \( \varphi(0) = 0 \), we also have \( \varphi(L) = 0 \) for some \( L \). Find a formula for \( L \) expressed in the maximal value \( \varphi_\text{max} \).

4. For a suitable \( 0 < a < 1 \), work through the details for the case \( f(u) = u(1-u)(u-a) \).

**Exercise 2** For the budworm model with dispersion, compute the integral

\[ F(u; R) = \int_0^R f(t; R) dt. \]

Which are its extreme points? Show that \( C'(R) > 0 \) and that when \( R \) is close to \( R_2 \) we have \( C(R) > 0 \) and when \( R \) is close to \( R_1 \) we have \( C(R) < 0 \).

**Exercise 3** Consider the model in the example above on the spread of rabies. Show that the system defining the travelling waves is equivalent to

\[
\begin{cases}
\frac{df}{dt} = c(r \ln g - f' - g + 1) \\
\frac{dg}{dt} = f g / c
\end{cases}
\]

with the boundary conditions \( f(\infty) = 0, \quad g(\infty) = 1 \). Determine the equilibria and their stability and conclude that a travelling wave solution can only occur for \( c \geq 2\sqrt{1-r} \).