The task is to write a short (around five pages) report on the topic below. The purpose of your report is to explain the dynamics of the situation to a friend interested in ecology but only with more basic mathematical background (but who easily understands what you explain). Having said that, there are some mathematical calculations to be done in the project.

Below you will find a list of exercises to carry out in order to get acquainted with the material.

**Introduction**

In the north woods of Canada the balsam trees have a competitive advantage over the birches, in competing for sun light and nutrition. With time the balsam trees would outcompete the birch tree if there was no spruce budworm. The budworm harm the balsam trees but not the birch: when there is an outbreak of budworms the balsam trees gets defoliated, which with time leads to the death of the larvae. A cycle takes around 4 years. During that time the birch take advantage of the situation and increase their numbers. After some time, however, the balsam trees come back, which leads to an increase in budworm numbers and the cycle restarts. A total cycle takes between 40 and 100 years. There is a huge economic interest in the balsam trees, so understanding this ecological interest is of great importance.

There are different time scales in this problem:

1. the budworm density can increase hundredfold and more in months
2. The larvae are eaten by birds, which can change their diet on the same
time scale
3. A defoliated tree can replace its foliation in 7–10 years
4. The life span of the trees involved is of size 100-150 years, but generation
time are a few years.

For the exercises we introduce the following three variables:

\[ B(t) = \text{the spruce budworm density} \]
\[ S(t) = \text{the total surface area of balsam trees} \]
\[ E(t) = \text{the energy reserve of balsam trees} \]

**Exercises**

1. Explain the biological assumptions behind the following model:

\[
\begin{align*}
B' &= r_B B(1 - \frac{B}{\alpha S}) - \frac{aB^2}{(\beta S)^2 + B^2}, \\
S' &= r_S S(1 - \frac{S}{K_{SE}}), \\
E' &= r_E E(1 - \frac{E}{K_E}) - \frac{PB}{S}
\end{align*}
\]

The factor \( K_{SE}/E \) is inserted into the second equation, because \( S \) does
not inevitably increase under conditions of stress – surface area may
decline through the death of branches or even whole trees.

2. Since the time-scale for the budworm density is much faster than that
for the other two variables, assume these are fixed and derive the fol-
lowing nondimensional version of the budworm density:

\[
u' = u(1 - u/Q) - \frac{1}{R} \frac{u^2}{1 + u^2}.
\]

Here \( Q \) depends only on properties of the budworm and their predators,
but not on tree quality, whereas \( R \) depends is proportional to the tree quality.
3. This model has one or three nonzero steady states, depending on $Q$ and $R$. Show that the region in the $RQ$-plane which gives three nonzero steady states is bounded by the curve

$$R(t) = \frac{2t^3}{(1+t^2)^2}, \quad Q(t) = \frac{2t^3}{t^2 - 1}, \quad t > 1.$$  

Make a graph showing the region.

4. Vary the parameters and explain why the outbreak occurs and what terminates it, and why the cycle repeats itself.

5. Next we hold the budworm density constant. Investigate the behaviour of the $(S,E)$-system as we change $B$ from a low to a large value, and interpret the results in biological terms.