Hand-in three
Critical outbreak patch sizes

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The task is to write a short report on the topic below. You should imagine that you have got a question by a biology friend to whom you are to explain the results of your analysis.

Please note that the report to be written is not expected to be mathematically solid. Instead you are expected to do some initial analysis from which to draw some plausible conclusions. Ideally these should be substantiated by some simulations for illustration.

Introduction

Consider a species which is described by an equation

\[ N' = f(N), \]

where \( f(0) = 0 \), and that this is an unstable equilibrium. Under those assumptions, if a small population come from outside, their numbers will increase.

However, if we also assume that the animals move around in a random way, modeled by a diffusion process, and that this small population appear only in a localized region \((0, L)\), \textit{outside} which the environment is so hostile that survival is impossible, the situation becomes more complicated. In such a case some individuals will be lost at the boundary, and if \( L \) is very small it may well be that the population cannot survive the losses on the boundary. The project is first to find a critical patch size \( L^* \) such that the population cannot sustain itself against boundary losses if the patch size is less than \( L^* \) but can maintain itself indefinitely if the patch size is greater than \( L^* \). The

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learnings from this should be used to address the more complicated problem of how to determine the patch size that is necessary to sustain an outbreak of the spruce budworm.

**Exercises**

Below you will find a list of exercises to carry out in order to get acquainted with the material.

When you carry out these exercises, do the first four first and write up the report on that first. Only when you have full control over this part should you continue with the budworm model.

1. Justify the nondimensional version of the problem:

   \[ u_t = u_{xx} + f(u), \quad u(0,t) = u(L,t) = 0. \]

2. Analyse the linear model with \( f(u) = ku \). Find all solutions of the form \( u(x,t) = a(t)b(x) \). Fourier analysis shows that the general solution can be written as an (infinite) superposition of such solutions. Use that to find a critical patch size for this case. Deduce a condition for the general model under which \( u^* = 0 \) will be stable for the model with diffusion.

3. Now consider the logistic model with \( f(u) = u(1-u) \). Describe, for given \( L \), as far as you can, when there is a steady state solution \( v(x) \) (not identically zero) to the problem. If \( v_m \) is its maximum, show that

   \[ L = \sqrt{2} \int_0^{v_m} \frac{dv}{\sqrt{F(v_m) - F(v)}}, \quad F(v) = v^2/2 - v^3/3. \]

   Describe \( v_m \) as a function of \( L \).

4. Can you justify from the above, and perhaps some simulations, that for the logistic model there is a \( L^* \) (which you should determine) such that

   (a) if \( L < L^* \), then \( u(x,t) \to 0 \) as \( t \to \infty \),

   (b) if \( L > L^* \), then \( u(x,t) \to v(x) \) as \( t \to \infty \) for some function \( v(x) > 0 \) in \( (0, L) \)?
Strictly speaking we need more mathematics here, but try to justify the statements as far as you can. The relation
\[ u_t - u_{xx} - u = -u^2 \leq 0 \]
may be used for a heuristic argument. Thus the strip is a stable refuge if its width is greater than \( L^* \) and not a refuge if \( L < L^* \).

5. Next consider the spruce budworm model

\[ u_t = u_{xx} + f(u; R), \quad f(u; R) = u(1 - u/Q) - \frac{u^2}{R(1 + u^2)}. \]

But this time we want to calculate a critical length for the strip to support an outbreak. We know that for given \( Q > 3^{3/2} \) there are two values \( R_1(Q) > R_2(Q) \) for \( R \) such that the line \( v = R(1 - u/Q) \) is tangent to the curve \( v = u/(1 + u^2) \).

Draw typical graphs \( v = F(u; R, Q) \) (where \( F \) is the primitive function to \( f \) which is zero when \( u = 0 \)) for the cases \( R < R_2(Q) \), \( R_2(Q) < R < R_1(Q) \) and \( R > R_1(Q) \). Use this to plot the function

\[ h(v_m) = \sqrt{2} \int_0^{v_m} \frac{dv}{\sqrt{F(v_m; R, Q) - F(v; R, Q)}}. \]

In the analysis there is a particular \( R = \tilde{R}(Q) \) which becomes important. What is its significance?

6. What have we learnt from this about what critical length is required to support an outbreak?