Students may use the provided formula sheet and a pocket calculator. All solutions should be properly justified.

1. Put \( f = x^4 - 4x^2 + y^4 + y^2 + 5 \) and let \((x, y) = (1/2, 1/2)\) be the initial point for a multidimensional search minimizing \( f \).
   a) Find all stationary points of \( f \) and show that a global minimum exists.
   b) Compute the first search directions for the steepest descent method and Newton’s method respectively.
   c) Replace Newton’s method by the modified Newton method with \( \epsilon = 10 \) and compute the corresponding direction \( d \).

2. a) Give the definition of a convex set in \( \mathbb{R}^n \).
   b) Prove that \( S, T \) convex sets \( \Rightarrow S \cap T \) convex.
   c) Is the function \( f(x, y) = e^{\sqrt{x^2+y^2}} + 2x + e^y \) convex on \( \mathbb{R}^2 \)?
   d) Is the set \( S = \{(x, y, z) \in \mathbb{R}^3; z \geq e^{\sqrt{x^2+y^2}} + 2x + e^y \} \) convex?

3. Consider the LP problem to minimize \(-x_1 - x_2 - x_3 - 3x_4 - 3x_5\) subject to
   \[
   \begin{align*}
   x_1 &+ 2x_2 &+ 3x_3 &+ 3x_4 &+ 5x_5 &\leq 7 \\
   x_1 &+ 3x_2 &+ 4x_3 &+ 5x_4 &+ 6x_5 &\geq 10 \\
   &+ x_3 &+ x_4 &\quad &\quad &\quad &2 \\
   x_i &\geq 0 &\text{for all } i.
   \end{align*}
   \]
   a) Compute a basic feasible solution using \( x_1, x_2 \) and \( x_4 \) as basic variables.
   b) Solve the problem. Is the solution unique?
   c) State the dual problem.
   d) Find a solution to the dual problem.
4. a) Consider the LP problem
\[
\min_{x \in S} c^T x \quad S: Ax \geq b, \quad x \geq 0.
\]
Let \(T\) denote the set of feasible points for the dual problem.
Prove that \(\hat{x} \in S\) and \(\hat{y} \in T\) satisfy the CSP equations if and only if \(c^T \hat{x} = \hat{y}^T b\).

b) Prove that
\[
\begin{align*}
    x_1 + x_2 + x_3 & \geq 0 \\
    x_1 + 3x_2 + 10x_3 & \geq 0 \Rightarrow 3x_1 + 5x_2 + 10x_3 \geq 0 \\
    -x_1 + x_2 + 5x_3 & \geq 0
\end{align*}
\]

5. Solve the problem
\[
\begin{align*}
\text{Minimize} & \quad x_1 + 2x_2 + 3x_3 \\
\text{subject to} & \quad x_1^2 + 2x_2^2 + 3x_3^2 \leq 9 \\
& \quad x_2 \geq 0.
\end{align*}
\]
using the Kuhn-Tucker theory. Do not forget to motivate why the global minimum exists.

6. Use the dual problem to find the global minimum of the function \(f = x^T x\) on \(\mathbb{R}^n\)
under the non-empty constraints \(Ax = b\) where \(b \in \mathbb{R}^m\), with \(m < n\), and \(A\) is an \(m \times n\) matrix such that the constraint set \(\{x : Ax = b\}\) is non-empty for each \(b\).

\textit{GOOD LUCK!}