Problem 12.20

Let \( A \) be an \( n \times n \) matrix.
Prove that \( A = e^S \) where \( S \) is real and skew-symmetric, i.e. \( S^* = S^T = -S \) if and only if \( A \) is real orthogonal and \( \det(A) = 1 \).

Solution

1) We first prove that if \( S \) is real and skew-symmetric, then \( A = e^S \) is real, orthogonal and \( \det(A) = 1 \).
Since \( A = \sum_{k=0}^{\infty} \frac{1}{k!} S^k \) where all terms are real, it is clear that \( A \) is real. The orthogonality follows from
\[
A^T A = e^{S^T} e^S = e^{-S} e^S = e^{-S+S} = e^0 = I.
\]
Taking the determinant we see that
\[
1 = \det (I) = \det (A^T A) = \det A \cdot \det A^T = (\det A)^2.
\]
Consequently \( \det A = \pm 1 \), but on the other hand
\[
\det(A) = \det \left( e^{S/2} e^{S/2} \right) = (\det e^{S/2})^2 > 0,
\]
so that \( \det A = 1 \).

2) It remains to show that if \( A \) is real, orthogonal with determinant 1 then \( A = e^S \) with \( S \) real and skew-symmetric. Here the requirement that \( S \) be real complicates the proof.
Consider first the special case \( n = 2 \) and
\[
A = A_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},
\]
where \( t \) is real. (This corresponds to a rotation by \( t \) radians.) With
\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
we have \( J^2 = -I \) and hence \( J^{2l} = (-1)^l I \), \( J^{2l+1} = (-1)^l J \) and
\[
e^{tJ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k = \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l}}{(2l)!} I + J \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l+1}}{(2l+1)!} = I \cos t + J \sin t,
\]
so that $A = e^{tJ}$ with $tJ$ real and skew-symmetric.

We now consider the general case. We shall see below that there exists an orthogonal matrix $O$ such that

$$A = OBO^T,$$

(1)

where

$$B = \text{blkdiag}(A_{t_1}, A_{t_2}, ..., A_{t_k}, I_l),$$

where $2k + l = n$. Then

$$\tilde{S} = \text{blkdiag}(t_1 J, t_2 J, ..., t_k J, 0_l)$$

is real, skew-symmetric and

$$A = Oe^{\tilde{S}}O^T = e^{S},$$

where $S = O\tilde{S}O^T$ is real and skew-symmetric.

3) It remains to verify (1). Since $A$ orthogonal and real, it is unitary and hence normal.

(Thus there exists an orthogonal basis of eigenvectors and a corresponding unitary matrix $U$ such that

$$A = UDU^*$$

where the $D$ is diagonal. Its diagonal elements, $d_j$, are eigenvalues of $A$. Since $A$ is unitary they have modulus 1 so that $d_j = e^{it_j}$ for some real $t_j$. Thus $D = e^{iF}$ where $iF$ is skew-symmetric, but unfortunately not real. To handle this we shall use that the non-real eigenvalues of $A$ occur as conjugated pairs.)

Since $A$ is real we have

$$p_A(\lambda) = p_+(\lambda)p_-(\lambda)p_0(\lambda),$$

where

$$p_+(\lambda) = (\lambda - z_1)...(\lambda - z_m)$$

where $z_1, ..., z_m$ are the eigenvalues with positive imaginary part, counted with multiplicities,

$$p_-(\lambda) = (\lambda - \bar{z_1})...(\lambda - \bar{z_m})$$

and

$$p_0(\lambda) = (z - \alpha_1)...(z - \alpha_k),$$
with $\alpha_j \in \mathbb{R}$, $1 \leq j \leq k$.
As $A$ is unitary, all the eigenvalues have modulus (absolutbelopp) one, so that
$z_j = e^{it_j}$ with $0 < t_j < \pi$, and

$$p_0(\lambda) = (\lambda - 1)^{m_+} (\lambda + 1)^{m_-},$$

with $m_- + m_+ = k = n - 2m$. Furthermore $m_-$ (the multiplicity of $-1$}

as an eigenvalue) must be even ($m_- = 2r$) since

$$1 = \det A = \lambda_1 \lambda_2 \ldots \lambda_n = (-1)^{m_-}.$$

(Note that $z_j \overline{z_j} = |z_j|^2 = 1$ for $1 \leq j \leq m$.)

Being normal $A$ has an ON-basis of eigenvectors $u_1, \ldots, u_n$. We may assume

that these are numbered so that

$$Au_{2j-1} = e^{it_j}u_{2j-1}, \quad 1 \leq j \leq m$$
$$Au_{2j} = e^{-it_j}u_{2j}, \quad 1 \leq j \leq m$$
$$Au_j = -u_j, \quad 2m + 1 \leq j \leq 2m + 2r$$
$$Au_j = u_j, \quad j > 2m + 2r.$$

We may also assume that $u_{2j} = \overline{u_{2j-1}}$ for $1 \leq j \leq m$, since if $Au = zu$ taking

complex conjugates gives

$$A\overline{u} = \overline{Av} = z\overline{u} = \overline{zu}.$$

(Here it is important that $A$ is real.)

Furthermore, for $j > 2m$ the eigenvectors $u_j$ may be chosen real since they

are obtained by solving the systems

$$(A + I)X = 0$$

which have real coefficients.

We now introduce a new set of $n$ real vectors via

$$e_{2j-1} = \sqrt{2} \Re u_{2j-1} = \frac{1}{\sqrt{2}}(u_{2j-1} + u_{2j}), \quad 1 \leq j \leq m$$
$$e_{2j} = \sqrt{2} \Im u_{2j-1} = \frac{1}{i\sqrt{2}}(u_{2j-1} - u_{2j}), \quad 1 \leq j \leq m$$
$$e_j = u_j, \quad j > 2m.$$
They have norm one. For \( e_j \) with \( j > 2m \) this is obvious. For \( e_{2j-1} \) with \( j \leq m \) this is seen through

\[
(e_{2j-1} | e_{2j-1}) = \frac{1}{2} (u_{2j-1} + u_{2j} | u_{2j-1} + u_{2j})
\]

\[
= \frac{1}{2} ((u_{2j-1} | u_{2j-1}) + 2 \text{Re}(u_{2j-1} | u_{2j}) + (u_{2j} | u_{2j})
\]

\[
= \frac{1}{2} (1 + 0 + 1) = 1.
\]

and for \( e_{2j} \) through a similar calculation.
Moreover, the vectors \( e_j \) are orthogonal, i.e.

\[
(e_j | e_k) = 0 \quad \text{for } j < k.
\]

If there is no \( l \leq m \) such that \( j = 2l - 1 \) and \( k = 2l \) then this is evident, since \( e_j \) and \( e_k \) then consist of different sets of the orthonormal vectors \( u_j \).
In the remaining case we get

\[
(e_{2l-1} | e_{2l}) = \frac{1}{2i} (u_{2l-1} + u_{2l} | u_{2l-1} - u_{2l})
\]

\[
= \frac{1}{2i} ((u_{2l-1} | u_{2l-1}) - (u_{2l-1} | u_{2l}) + (u_{2l} | u_{2l-1}) - (u_{2l} | u_{2l})
\]

\[
= \frac{1}{2i} (1 - 0 + 0 - 1) = 0.
\]

Finally, for \( 1 \leq j \leq m \) we have

\[
A e_{2j-1} = \frac{1}{\sqrt{2}} (e^{it_j} u_{2j-1} + e^{-it_j} u_{2j}) = \cos t_j e_{2j-1} - \sin t_j e_{2j}
\]

\[
A e_{2j} = \frac{1}{i\sqrt{2}} (e^{it_j} u_{2j-1} - e^{-it_j} u_{2j}) = \sin t_j e_{2j-1} + \cos t_j e_{2j}
\]

whereas \( A e_j = -e_j \) for \( 2m + 1 \leq j \leq 2m + 2r \) and \( A e_j = e_j \) for \( j > 2m + 2r \).
Thus \( O = [e_1 \ldots e_n] \) is (real) orthogonal and

\[
B = O^T A O = \text{blkdiag}(A_{t_1}, \ldots, A_{t_m}, -I_2, I_{m_+})
\]

\[
= \text{blkdiag}(A_{t_1}, \ldots, A_{t_m}, A_\pi, \ldots, A_\pi, I_{m_+}),
\]

which has the desired form. Here we used that

\[
-I_2 = \text{blkdiag}(-I_2, \ldots, -I_2) = \text{blkdiag}(A_\pi, \ldots, A_\pi).
\]