Lecture 5: Compactness

Definition: \( A \subset V \) is bounded if
\[
\sup_{x \in A} \| x \| < \infty.
\]

Definition Let \( V \) be a normed space. We say that \( M \subset V \) has the \textbf{Bolzano-Weierstrass property (BW)} if every sequence \( \langle x_n \rangle \) in \( M \) has a convergent subsequence \( \langle x_{n_k} \rangle \).

Remark: A subsequence \( \langle x_{n_k} \rangle \) of a given sequence \( \langle x_n \rangle_{n \in I} \) is obtained by choosing an infinite subset \( \tilde{I} \subset I \) and then a monotone enumeration (uppräkning) \( \langle n_k \rangle_{k=0}^{\infty} \) of the elements in \( \tilde{I} \).

Example A set with three elements \( M = \{a_1, a_2, a_3\} \) has (BW), because if \( \langle x_n \rangle \subset M \) is a sequence, that at least one of the three sets
\[
N_j = \{ n \in \mathbb{N} : x_n = a_j \}
\]
has infinitely many elements.

The sequence \( a_1, a_2, a_3, a_1, a_2, a_3, a_1, \ldots \) shows that there need not be convergence before the selection, and that there may be several possible limit points.
Theorem 1 (≈ Bolzano-Weierstrass) Let $M \subset \mathbb{R}^d$ be closed and bounded. Then it has (BW).

Proof of Theorem 1: (in the case $d = 3$) Choose a closed cube $K$ which contains $M$. Denote the side-length of $K$ by $a$. We may split $K$ into $2^3$ subcubes with sidelength $2^{-1}a$. For at least one of these, denoted $K_1$, it is true that $x_n \in \overline{K_1}$ for infinitely many $n$. Let $n_1$ be the first of these, and then split $K_1$ into $2^3$ subcubes with sidelength $2^{-2}a$. For at least one of these we have $x_n \in \overline{K_2}$ for infinitely many $n > n_1$. Choose $n_2$ as the smallest of these and then split $K_2$ into subcubes etc. In this way we obtain a sequence of cubes $\langle K_j \rangle$ with sidelengths $2^{-j}a$ such that $\overline{K_{j+1}} \subset \overline{K_j}$ and a subsequence $\langle x_{n_k} \rangle$ with $x_{n_k} \in \overline{K_k}$. In particular, we have

$$|x_{n_k} - x_{n_l}| \leq C2^{-\min(k,l)}a$$

so that this subsequence is Cauchy. As $\mathbb{R}^3$ is complete, the subsequence has a limit, $x$, which must belong to $M$ since $M$ is closed. $\square$
Theorem 2 Assume that $M \subset V$ has (BW) and that $f : M \rightarrow \mathbb{R}$ is continuous. Then $f$ is bounded on $M$ and there exist $\tilde{y}, \tilde{z} \in M$ such that

$$f(\tilde{y}) = \sup_{x \in M} f(x) \quad f(\tilde{z}) = \inf_{x \in M} f(x).$$

Proof: If $f$ is unbounded on $M$, there there is a sequence $\langle x_n \rangle \subset M$ with $|f(x_n)| > n$. According to (BW) there is a subsequence $x_{n_k} \rightarrow \tilde{x} \in M$. We obtain

$$|f(\tilde{x})| = \lim_{k \rightarrow \infty} |f(x_{n_k})| \geq \lim_{k \rightarrow \infty} n_k = +\infty,$$

which contradicts the fact that $f$ has a (finite) value in $\tilde{x}$. 
Thus $m = \sup_{x \in M} f(x) < \infty$, and (by def of sup) there is a sequence $\langle x_n \rangle$ with $f(x_n) > m - 1/n$. Another application of (BW) gives a subsequence $x_{n_k} \to \tilde{y}$ with

$$f(\tilde{y}) = \lim_{k \to \infty} f(x_{n_k}) = m.$$ 

Consequently, the function $f$ assumes its supremum and, since this is true for $-f$ as well, its infimum. \hfill \qed

**Theorem 3** Let $M \subset V$ have (BW). Then $M$ is closed and bounded.

**Proof:** That $M$ is bounded follows from the fact that the continuous function $x \mapsto \|x\|$ is bounded on $M$ by Theorem 2.

It remains to show that $\bar{C}M$ is open. Let $z \notin M$. By Theorem 2, the function $x \mapsto \|x - z\|$ has a smallest value, $\delta$, on $M$. We must have $\delta > 0$ and thus $B(z, \delta/2)$ is an open ball centered at $z$ which does not intersect $M$. \hfill \qed
Known from Calculus in Several Variables:

**Definition**  $M \subset \mathbb{R}^d$ is called **compact** if it is closed and bounded.

**Definition**: We say that $M \subset V$ is **compact** if $M$ has (BW).

The following example shows that the conditions that $M$ be closed and bounded would not be sufficient in Theorem 2.

**Example** Let $B = l^2$ and set

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j}x(j)^2.$$ 

Then, by the Cauchy-Schwarz inequality,

$$|f(x) - f(y)| \leq \sum_{j=0}^{\infty} |x(j)^2 - y(j)^2| \leq \|x + y\| \|x - y\|,$$

and consequently $x \mapsto f(x)$ is continuous.
Now, consider the unit sphere $S_1 = \{x \in l^2 : ||x||^2_2 = \sum_{j=1}^{\infty} x(j)^2 = 1\}$. This is a closed bounded set such that $f(x) > 0$ when $x \in S_1$. The function $f$ does not have a smallest value on $S_1$ since $x_k(j) = \delta_{jk}$ gives $f(x_k) = 1/k$, which tends to zero as $k \to \infty$. We also see that the function
\[
g : S_1 \ni x \longmapsto \frac{1}{f(x)}
\]
is continuous but unbounded.

**Theorem 4** Let $f$ be a continuous map $V \longmapsto W$ where $V$ and $W$ are normed spaces. Then $f(M)$ is compact if $M$ is compact.

**Proof:** Consider a sequence $\langle y_n \rangle = \langle f(x_n) \rangle \subset f(M)$. Since $M$ is compact there is a convergent subsequence $x_{n_k} \to \tilde{x} \in M$. But then we have
\[
y_{n_k} = f(x_{n_k}) \to f(\tilde{x}) = \tilde{y} \in f(M).
\]
Example, showing that the unit sphere need not be compact when $\dim V = \infty$. Let $V = C(0, 2\pi)$ with norm given by $\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \, dt$. With $f_n(t) = e^{int}$ we have $\|f_n\| = 1$ for all $n \in \mathbb{N}$, and if $m \neq n$ we get

$$
\left\| f_n - f_m \right\|^2 = \frac{1}{2\pi} \int_0^{2\pi} (e^{-int} - e^{-imt})(e^{int} - e^{imt}) \, dt
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} (2 - 2 \cos (n-m)t) \, dt
$$

$$
= 2.
$$

Consequently there are no element in the sequence $\langle f_n \rangle$ which are less than $\sqrt{2}$ apart, and there is no convergent subsequence.

In a certain sense compactness may be seen as replacing completeness by a stronger condition.

**Theorem 5** Every compact set, $M$, is complete.

**Proof:** If $\langle x_n \rangle$ is Cauchy in $M$, then due to compactness it has a convergent subsequence $x_{n_k} \rightarrow \tilde{x}$. It remains to prove that the original sequence is convergent. Let $\epsilon > 0$. By the def of Cauchy sequence there is an $N = N(\epsilon)$ such that

$$
\left\| x_m - x_{n_k} \right\| \leq \epsilon \quad \text{om} \quad m, n_k \geq N.
$$

If we let $k \rightarrow \infty$ we obtain

$$
\left\| x_m - x \right\| \leq \epsilon \quad \text{om} \quad m \geq N.
$$

\[\square\]
Compactness and dimension

**Theorem 6** Let $V$ be a vector space with $\dim V < \infty$. Then all norms on $V$ are equivalent. In particular, any set which is open (closed) with respect to one norm, is open (closed) w r t all others.

**Lemma 1** Let $e_1, e_2, \ldots, e_n$ be linearly independent in a normed space $V$, and set $W = \mathbb{R}^n$, equipped with the 2-norm. Then the functions

$$F_e : W \ni x \mapsto x = \sum_{k=1}^{n} x_k e_k \in V$$

and $G_e : W \ni x \mapsto \| F_e(x) \|$ are continuous.

**Proof:** We have (by exercise)

$$\| x \| - \| y \| \leq \| x - y \|,$$

and hence, since $F_e$ is linear, it is sufficient to show that

$$\| F_e(z) \| \leq C_e \| z \|_2.$$

But this follows, since

$$\| \sum_{k=1}^{n} x_k e_k \| \leq \sum_{k=1}^{n} \| x_k e_k \| \leq \sum_{k=1}^{n} \| x_k \| \| e_k \| \leq C_e \| x \|_2,$$

where $C_e^2 = \sum_{k=1}^{n} \| e_k \|^2$, by Cauchy-Schwarz. \qed

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Proof of the theorem: Let $e_1, e_2 \ldots e_n$ be a basis for $V$. Then any vector $x \in V$ may be written
\[ x = F_e(x) = \sum_{k=1}^{n} x_k e_k. \]
It is easy to see that $\|x\|_e := \|x\|_2$ is a norm. It is sufficient to show that, for any norm $\|x\|_V$, there are constants $c_V, C_V > 0$ such that
\[ c_V \|x\|_e \leq \|x\|_V \leq C_V \|x\|_e, \]
and since $\|\lambda x\| = |\lambda| \|x\|$ for every norm, we may assume that
\[ x \in S^1 := \{x \in V : \|x\|_e = 1\} = F_e(S^{n-1}), \]
where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Thus, it is enough to show that
\[ 0 < c_V \leq G_e(x) \leq C_V \]
when $x \in S^{n-1}$. This, however, follows from Lemma 1 and Theorem 2 since $S^{n-1}$ is compact (and $F_e(x) = 0$ only for $x = 0$). \qed
Theorem 7 Let $B$ be a Banach space over $\mathbb{R}$ (or $\mathbb{C}$). Then the closed unit ball $B(1)$ in $B$ is compact if and only if $\dim V < \infty$.

Proof: If $\dim B = n < \infty$ and $e_1, e_2, \ldots, e_n$ is a basis, then it follows from Theorem 6 that

$$K = \{x \in \mathbb{R}^n : \sum x_k e_k \in B(1)\}$$

is closed and bounded. Consequently, it is compact which (by Theorem 4) implies that $B(1) = F_e(K)$ is compact.

If $\dim B = \infty$, so that $B$ has an infinite sequence of finitely dimensional subspaces $\langle B_j \rangle$ with $B_j \subset B_{j+1}$ and $\dim B_j = j$, then it follows from lemma 2, below, that $B(1)$ contains a sequence, $\langle x_n \rangle$ such that $\|x_{n_1} - x_{n_2}\| \geq 1$ if $n_1 \neq n_2$, i.e., without convergent subsequence. \qed

\begin{equation}
K = \{x \in \mathbb{R}^n : \sum x_k e_k \in B(1)\}
\end{equation}
Lemma 2 (F. Riesz) Let $B_1 \subset B_2 \subset \ldots$ be a strictly increasing sequence of finite-dimensional subspaces of a Banach space $B$. Then there are $x_j \in B_j$, $k \geq 1$, such that

$$\|x_j - x\| \geq \|x_j\| = 1 \quad \text{when} \quad x \in B_{j-1}.$$  

**Proof:** Choose $y_j \in B_j \setminus B_{j-1}$ ($B_0 = 0$.) Consider the function $g_j(y) = \|y - y_j\|$ on $B_{j-1}$. It is continuous, and consequently has a minimum on each ball $\|y\| \leq R$ (which is compact since $\dim B_{j-1} < \infty$). By (1), it follows however that

$$g_j(y) \geq \|y\| - \|y_j\|$$

which shows that $g_j(y) > g(0)$ outside the ball if the radius $R$ is sufficiently large. Hence there is a point $z_j \in B_{j-1}$ where $g_j$ attains its minimum.

In particular, we have

$$0 < \|z_j - y_j\| \leq \|x + z_j - y_j\| \quad \text{if} \quad x \in B_{j-1},$$

and we may choose $x_j = (y_j - z_j)/\|y_j - z_j\|$.
Compact subsets of $C(K)$ when $K$ is compact

In order to formulate the conditions that characterize compact subsets of $C(K)$, we introduce two new concepts.

**Definition** A set $F$ of real-valued functions on $M \subset V$, where $V$ is a $V$ normed space, is uniformly bounded if there exists a constant $C_F$ such that

$$|f(x)| \leq C_F \quad \text{for all } x \in M, f \in F.$$ 

It is equicontinuous if, for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{if } f \in F, x, y \in M, ||x - y|| < \delta.$$ 

**Definition** Let $M \subset V$ where $V$ is a normed space. If $\overline{M}$ is compact we say that $M$ is relatively compact. (In particular, every bounded set in $\mathbb{R}^n$ is relatively compact.)

(Equivalently: Every sequence in $M$ has a subsequence which has a limit in $V$, see Lemma 4.)

**Theorem 8 (Arzela-Ascoli)** Let $I = [a, b]$ be a compact interval. A subset $F$ of $C(I)$ is relatively compact (with respect to the supremum norm) iff $F$ is uniformly bounded and equicontinuous.

**Remark** We cannot replace relatively compact by compact: Assume that $F$ is a compact set and that $F \ni f_n \to f \in F$. Then $F_1 := F \setminus \{f\}$ is also uniformly bounded and equicontinuous but not closed—hence not compact.