Lecture 3: Completeness

It turns out that many useful theorems hold only for a subclass of normed vector spaces. This class is introduced by imposing an extra condition. This is analogous to the fact that several important theorems for continuous functions on $\mathbb{R}$ are not true if we replace $\mathbb{R}$ by $\mathbb{Q}$.

Example 3.1: The function

$$f : \mathbb{Q} \ni x \mapsto x^3 - 6x + 5 \in \mathbb{Q}$$

is $C^1$ with derivative

$$f'(x) = 3(x^2 - 2).$$

On the “interval” $I = [0, 2] \cap \mathbb{Q}$ its derivative $f'$ grows from $f'(0) = -6$ to $f'(2) = 6$ but does not attain the intermediate value 0. This implies that $f$ cannot have an interior minimum on the closed, bounded set $I$ and, since $f(3/2) = -5/8$ but $f(0) = 5$ and $f(2) = 1$, there is no minimum at all.
Example 3.2: The real-valued polynomials in one variable, $\mathbb{R}[x]$, form a vector space, which we may equip with the norm

$$\|p\|_\infty = \max_{0 \leq x \leq 1} |p(x)|.$$ 

Now consider the polynomials

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$ 

We know that $p_n(x) \to e^x$, when $n \to \infty$ and $x \in [0, 1]$, and since all the terms grow with $x$ it follows that

$$\max_{0 \leq x \leq 1} |p_n(x) - e^x| = |p_n(1) - e^1| \to 0.$$ 

Consequently, $p_n \to e^x$ in $C[0, 1]$ with $\| \cdot \|_\infty$ but the limit function is not a polynomial. Although the elements in the sequence approach each other rapidly as $n$ grows, there is no limit in $\mathbb{R}[x]$. It is this kind of phenomena one wants to exclude.
In the case of sequences of real (or complex) numbers one can prove a criterion for convergence which uses the following concept.

**Definition:** The sequence \((a_j)\) is a Cauchy sequence (or Cauchy) if, for any \(\epsilon > 0\), there exists a \(N = N(\epsilon)\) such that

\[
|a_j - a_k| < \epsilon, \quad \forall j, k > N.
\]

This condition only involves elements of the sequence, and it is satisfied if \(a_j \to a\), because

\[
|a_j - a_k| \leq |a_j - a| + |a - a_k|.
\]
**Theorem** *(Cauchy’s Criterion for Convergence, SSp 1.12–13)* A sequence of real (or complex) numbers is convergent iff it is Cauchy.

This is not true for \( \mathbb{Q} \). Indeed, take \( a_j = \sum_{k=0}^{j} \frac{1}{k!} \in \mathbb{Q} \). We know that \( a_j \to e^1 \in \mathbb{R} \setminus \mathbb{Q} \) in \( \mathbb{R} \), so we have a Cauchy sequence and, since there can be at most one limit in \( \mathbb{R} \), there is no limit in \( \mathbb{Q} \).

Thus \( \mathbb{Q} \) is incomplete in the sense that there are Cauchy sequences that “converge out of \( \mathbb{Q} \”).

If we replace the modulus sign in the definition of Cauchy sequence by a norm sign, we get:
**Definition:** A sequence \((x_j)\) in a normed vector space \(V\) is a **Cauchy sequence** if, for any \(\epsilon > 0\), there is an \(N = N(\epsilon)\) s. t.

\[ \| x_j - x_k \| < \epsilon, \quad j, k \geq N. \]

**Definition:** A normed space is called **complete** if every Cauchy sequence is convergent. A complete normed space is called a **Banach space**.

**Example 3.3:** \(\mathbb{R}^n\) is complete if

\[ \| x \| = \| x \|_2 = \left( \sum_{k=1}^{n} |x(k)|^2 \right)^{1/2}. \]

To see this, we note that from \( |x_k(j) - x_l(j)| \leq \| x_k - x_l \| \) it follows that the \(n\) sequences formed by the coordinates are Cauchy in \(\mathbb{R}\). Since \(\mathbb{R}\) is complete there are numbers \(x(j), 1 \leq j \leq n\), such that \(x_k(j) \to x(j)\), when \(k \to \infty\), and consequently

\[ \| x_k - x \|^2 = \sum_{j=1}^{n} |x_k(j) - x(j)|^2 \to 0. \]
Example 3.4: $l^2(\mathbb{N})$ is complete. Indeed, for any fixed $j$, we have $|x_k(j) - x_l(j)| \leq \|x_k - x_l\|_2$ which, as above, implies convergence for each coordinate. We obtain a sequence $x = \langle x(j) \rangle$ such that $x_k(j) \to x(j)$ for each $j$, as $k \to \infty$. It is, however, not obvious that $x \in l^2(\mathbb{N})$. Nevertheless, for each $N \in \mathbb{Z}_+$, we have

$$\sum_{j=1}^{N} |x_k(j) - x_l(j)|^2 \leq \|x_k - x_l\|_2^2 < \epsilon^2,$$

if $k, l \geq K = K(\epsilon)$. Thus, if we let $k \to \infty$ we obtain

$$\sum_{j=1}^{N} |x(j) - x_K(j)|^2 \leq \epsilon^2, \quad \forall N,$$

which implies that $x - x_K \in l^2(\mathbb{N})$. From this it follows that $x = (x - x_K) + x_K \in l^2(\mathbb{N})$ and that

$$\|x - x_K\| \to 0$$

as $K \to \infty$. 

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Example 3.5: $C[0, 1]$ is complete if we use the norm $\| \cdot \|_\infty$. As we shall see the proof is analogous with the one above:

For fixed $a \in I = [0, 1]$ we have $|f_j(a) - f_k(a)| \leq \| f_j - f_k \|_\infty$ which implies that $\langle f_j(a) \rangle$ has a limit, which we denote $f(a)$.

To prove that $a \mapsto f(a)$ is continuous, we shall show that for $a_0 \in I$ and $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(a) - f(a_0)| < \epsilon \quad \text{if} \quad a \in I \quad \text{and} \quad |a - a_0| < \delta.$$

By assumption there is a $N$ such that $\| f_j - f_k \|_\infty < \epsilon/3$ if $j, k \geq N$. Consequently, for every $a \in I$,

$$|f_j(a) - f_k(a)| < \epsilon/3, \quad \text{if} \quad j, k \geq N.$$

Letting $j \to \infty$ we obtain $|f(a) - f_N(a)| \leq \epsilon/3$, independently of $a$. Because the function $f_N$ is continuous we may conclude with the estimate

$$|f(a) - f(a_0)| < |f(a) - f_N(a)| + |f_N(a) - f_N(a_0)| + |f_N(a_0) - f(a_0)|.$$
**Exempel 3.5:** $C[-1, 1]$ equipped with the norm $|| \cdot ||_2$ is *not* complete. To verify this we study $f_n(x) = g(nx)$ where (draw a picture)

$$g(x) = \begin{cases} \frac{x}{|x|}, & |x| \geq 1, \\ x, & |x| < 1. \end{cases}$$

We first check that $\langle f_n \rangle$ is Cauchy. This follows from (assume $l > k$)

$$|| f_k - f_l ||^2_2 = \int_{|x| < 1/k} |f_k(x) - f_l(x)|^2 \, dx \leq \frac{2}{k}.$$ 

We now assume that $f_n \rightarrow f \in C[-1, 1]$ and choose a $\delta > 0$. If $n > 1/\delta$, then $|f_n(x)| = 1$ for $|x| > \delta$ and hence

$$|| f_n - f ||^2_2 \geq \int_{-\delta}^{-1} |1 - f(x)|^2 \, dx + \int_{\delta}^{1} |1 - f(x)|^2 \, dx.$$ 

Letting $n \rightarrow \infty$ we see that $f(x) = 1$ for $x > \delta$ and $f(x) = -1$ for $x < -\delta$. Since $\delta > 0$ is arbitrary, this implies that $f(x) = 1$ for $x > 0$ and $f(x) = -1$ for $x < 0$, i.e. $f \notin C[-1, 1]$. 

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In a same manner that the real numbers are introduced to make Cauchy sequences in \(\mathbb{Q}\) convergent, one can, for any normed space \(V\), use an abstract procedure to construct a Banach space \(B\) such that \(V\) may be identified with a dense subset in \(B\). The space \(B\) is essentially unique, and is called the completion of \(V\).

**Example 3.6:** The completions of \(C[0, 1]\) w.r.t. the norms \(\|\cdot\|_1\) and \(\|\cdot\|_2\), respectively, are denoted \(L^1[0, 1]\) and \(L^2[0, 1]\). \((L^1[0, 1]\) may be identified with the space of all (equivalence classes of) Lebesgue integrable functions on \([0, 1]\) (two functions being equivalent if they are equal except on a 'negligible set'). The space \(L^2[0, 1]\) may be described is similar way.)

**Sats SSp 1.14** A normed space \(V\) is complete if it has the property that a series, \(\sum_{k=0}^{\infty} a_k\), in \(V\) converges whenever the numeric series \(\sum_{k=0}^{\infty} \|a_k\|\) converges.

Such series are called absolutely convergent.

**Examples:** Since \(\mathbb{R}\) is complete, a numerical series converges if it converges absolutely. Since \(C[0, 1]\), with the supremum norm, is complete, we obtain that \(\sum_{k=1}^{\infty} u_k(x)\) converges in \(C[0, 1]\) if \(\sum_{k=1}^{\infty} \| u_k \|_\infty\) converges. This is the Weierstrass \(M\)-test.