Two norms (\(\| x \|\) and \(\|\| x \|\|\), say) on a vector space are equivalent if there are constants \(C > c > 0\) such that

\[ c\| x \| \leq \|\| x \|\| \leq C\| x \|. \]

This implies that a sequence is convergent wrt (with respect to) the first one iff (if and only if) it is convergent wrt the second one, since ....

Later we will see that if \(\dim V < \infty\), then all norms on \(V\) are equivalent. However: If

\[ f_n(x) = \begin{cases} 
1 - nx, & \text{om } x \leq 1/n, \\
0, & \text{om } x > 1/n.
\end{cases} \]

it is easily seen that \(\| f_n - 0 \|_1 \to 0\) but \(\| f_n - 0 \|_{\infty} = 1\), and thus these norms on \(C[0, 1]\) are not equivalent.
Open and closed sets

Many notions from $\mathbb{R}^3$ have natural generalizations:

Let $V$ be a normed space.

**Open ball:** $B(a, r) = \{x \in V : \|x - a\| < r\}$.

We say that $O \subset V$ is **open** if, for every $x \in O$, there is a $\epsilon = \epsilon(x) > 0$ such that $O$ contains the ball $B(x, \epsilon)$.

We say that $M \subset V$ is **closed** if $\mathbb{C}M = V \setminus M$ is open.

**Example S:** Let $V = C[0, 1]$ with the supremum norm, $\|f\|_{\infty}$. Then the subset

$$M = \{f \in V : \int_0^1 f(t) \, dt = 0\}$$

is closed, since ...

In the same manner the definitions of **inner point**, **boundary point**, **boundary** etc carry over.
Let $M \subset V$. A point $x \in V$ is said to be a point of accumulation (hopningspunkt) for $M$ if there is a sequence in $M \setminus \{x\}$ which converges to $x$.

Theorem 1.5 $M \subset V$ is closed if and only if $M$ contains all its points of accumulation. The closed hull (slutna höljet), $\overline{M}$, of $M \subset V$ is the union of $M$ and the set of all its points of accumulation. It is the smallest closed set which contains $S$. 
**Def** Let $V$ and $W$ be normed spaces. A mapping (funktion) $F : V \rightarrow W$ is said to be continuous if

$$x_m \rightarrow x \implies F(x_m) \rightarrow F(x).$$

If $O \subset W$ and $F : V \rightarrow W$ we set

$$F^{-1}(O) = \{x \in V : F(x) \in O\}.$$

**Theorem 1.11** The mapping $F : V \rightarrow W$ is continuous if and only if

$$O \subset W \text{ öppen } \implies F^{-1}(O) \text{ öppen.}$$

Example: Let $V$ be as in the example above. Then

$$U = \{f \in V : \left| \int_0^1 f(t) \, dt \right| < 1/3\}$$

is open.
If $S \subset T \subset V$ is such that $T \subset \overline{S}$ we say that $S$ is **dense (tät)** in $T$. This means (Theorem 4.49) that, for any $x \in T$, there exists a sequence in $S$ which converges to $x$, i.e. any element of $T$ may be approximated arbitrarily (godtyckligt) well by elements in $S$.

**Example** $\mathbb{Q}$ is dense in $\mathbb{R}$. The polynomials are dense in $C[a, b]$ (Weierstrass’ approximation theorem, SSp A.3). The set $C^\infty[0, 1]$ is dense in $C[0, 1]$, with the supremum norm, as now shall see (Appendix A.4).
Regularization

1. Extend \( f \) to obtain a function \( \tilde{f} \in C_0(\mathbb{R}) \).

2. Set

\[
\chi(t) = \begin{cases} 
  e^{-1/t} & t > 0 \\
  0 & t \leq 0 
\end{cases}
\]

Then the function \( \chi \in C^\infty(\mathbb{R}) \) and \( \psi(t) = \chi(1 + t)\chi(1 - t) \in C_0^\infty(\mathbb{R}) \setminus \{0\} \).

3. Set \( \phi(t) = \frac{1}{C} \psi(t) \), where \( C = \int \psi(t) \, dt \) and \( \phi_n(t) = n\phi(nt) \). Then

(a) \( \phi_n(t) \geq 0 \) and \( \int \phi_n(t) \, dt = 1 \).

(b) \( \phi_n(t) = 0 \) if \( |t| \geq 1/n \).

4. Set \( \tilde{f}_n(t) = \tilde{f} * \phi_n(t) \). Then

\[
\sup_{t \in [0,1]} |\tilde{f}(t) - \tilde{f}_n(t)| \rightarrow 0, \quad n \rightarrow \infty,
\]

and \( \tilde{f}_n \in C^\infty \). 

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