Suggestions for exercise 5.4 and some pre-calculated stuff

We study the problem:
\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} + \begin{bmatrix}
zx + x^3 + xy^2 \\
y + y^3 + z + x^2 y \\
0
\end{bmatrix}.
\]

Where \(z\) is actually a parameter. For \(z = 0\) we have a Hopf bifurcation. We seek an almost-identity coordinate transformation:
\[
x = u + h_1(u, v, z)
y = v + h_2(u, v, z)
\]
where \(h_i(0) = h'_i(0) = 0\). For small \(h\) this is just the identity. We want to choose \(h\) such that the resulting equation in \(u, v\) becomes as simple as possible.

The old equation was:
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + k(x, y, z)
\]

We call \(A\) to the linearization matrix, and want a new equation of the form:
\[
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} = A \begin{bmatrix}
u \\
v
\end{bmatrix} + F(u, v, z)
\]

Rewrite \((\dot{x}, \dot{y})\) using the coordinate transformation:
\[
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
A \begin{bmatrix}
u \\
v
\end{bmatrix} + F(u, v, z)
\end{bmatrix}
\]

\(h_{im}\) is taken to be a homogeneous polynomial of degree \(m = 2, 3, \ldots\) which we adjust in such a way that the corresponding \(F_{im}\) gets simple (zero if possible). The equation above is truncated to the corresponding degree \(m\). In this way, we construct the solution \(h\) (and \(F\) in the new equation) stepwise after \(m\). We can rewrite the equation as:
\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} = \begin{bmatrix}
k_1 \\
k_2
\end{bmatrix} + A \begin{bmatrix}
h_1 \\
h_2
\end{bmatrix} - \frac{\partial(h_{12})m}{\partial u, v} A \begin{bmatrix}
u \\
v
\end{bmatrix}
\]

Now we solve the equation for each power \(m\). We just do \(m = 2, 3\):
Propose
$$
\begin{align*}
    h_1(u,v,z) &= \alpha_1 u^2 + \alpha_2 v^2 + \alpha_3 z^2 + \alpha_4 uv + \alpha_5 uz + \alpha_6 vz \\
    h_2(u,v,z) &= \beta_1 u^2 + \beta_2 v^2 + \beta_3 z^2 + \beta_4 uv + \beta_5 uz + \beta_6 vz \\
    F_1(u,v,z) &= a_1 u^2 + a_2 v^2 + a_3 z^2 + a_4 uv + a_5 uz + a_6 vz \\
    F_2(u,v,z) &= b_1 u^2 + b_2 v^2 + b_3 z^2 + b_4 uv + b_5 uz + b_6 vz
\end{align*}
$$

Rewriting the above equation it results in an algebraic equation for the coefficients $\alpha, \beta$ which are to be chosen according to the desired result, namely that the polynomial equation holds with simplest possible choice of coefficients $a, b$. The solution is:

$$
\begin{align*}
    a_1 u^2 + a_2 v^2 + a_3 z^2 + a_4 uv + a_5 uz + a_6 vz &= zu - (\beta_1 u^2 + \beta_2 v^2 + \beta_3 z^2 + \beta_4 uv + \beta_5 uz + \beta_6 vz) + \nu \left( \frac{\partial (h_1)_{m}}{\partial u} - \nu \frac{\partial (h_1)_{m}}{\partial v} \right) \\
    b_1 u^2 + b_2 v^2 + b_3 z^2 + b_4 uv + b_5 uz + b_6 vz &= zv + v^2 + (\alpha_1 u^2 + \alpha_2 v^2 + \alpha_3 z^2 + \alpha_4 uv + \alpha_5 uz + \alpha_6 vz) + \nu \left( \frac{\partial (h_2)_{m}}{\partial u} - \nu \frac{\partial (h_2)_{m}}{\partial v} \right)
\end{align*}
$$

If the polynomials in both sides are to be equal, we obtain a $12 \times 12$ algebraic equation system:

$$
\begin{pmatrix}
    0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \alpha_3 \\
    \alpha_4 \\
    \beta_1 \\
    \beta_2 \\
    \beta_3 \\
    \beta_4 \\
    \beta_5 \\
    \beta_6 \\
\end{pmatrix}
=
\begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3 \\
    a_4 \\
    b_1 \\
    b_2 \\
    b_3 \\
    b_4 \\
    b_5 \\
    b_6 \\
\end{pmatrix}
$$

Notice that the equation is separated in blocks: A $6 \times 6$ block, two $1 \times 1$ and a $4 \times 4$ block. The determinant of the coefficient submatrix for the first three blocks is non zero and hence there exists a unique solution, no matter which right hand side we have. We can then take coefficients $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ to be zero which in turn gives that $\beta_1 = \alpha_1 = 2\alpha_2 = -2/3$ and $\alpha_3 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = 0$ in $h$. The last block has zero determinant which means that we cannot find solutions $\alpha, \beta$ for whichever right hand side $a, b$ we want. We have to compromise. The trivial solution is always present, so we get that for $a_6 = b_5 = 0$ and $a_5 = b_6 = 1$ there exists the solution $\alpha_5 = \alpha_6 = \beta_5 = \beta_6 = 0$ (among many others). Summary: $F = (uz, vz)$ och $h = \left(-\frac{2}{3}u^3 - \frac{1}{3}v^2 - vz, 0\right)$. 

$m = 2$ 

\( m = 3 \): The same procedure, but harder. Now have all \( h \) and \( F \) degree 3 and hence 10 coefficients each. On the right hand side we compute \( k_3 \) using \((h_1, h_2)_2\) and we end up with a 20 \( \times \) 20 system. Let us write \( h_1 \) to check which part of the polynomial follows with each coefficient:

\[
h_1(u, v, z) = \alpha_1 u^3 + \alpha_2 v^3 + \alpha_3 z^3 + \alpha_4 u^2 v + \alpha_5 u^2 z + \alpha_6 uv^2 + \alpha_7 u w z + \alpha_8 u z^2 + \alpha_9 v^2 z + \alpha_{10} v z^2.
\]

There are now five blocks: 8 \( \times \) 8 and 4 \( \times \) 4 have zero determinant yielding the zero solution for zero right hand side (among others), which means a non trivial \( F \):

\[
\begin{pmatrix}
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
3 & 0 & 0 & -2 & 0 & 0 & -1 & 0 \\
0 & -3 & 2 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 3 & 0 & 0 & -2 \\
0 & 0 & 0 & 1 & 0 & -3 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{pmatrix}
= \begin{pmatrix}
a_1 - 1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 - 1 \\
b_1 \\
b_2 - 1 \\
b_3 - 1
\end{pmatrix}.
\]

Zero solution results in all \( \alpha, \beta \) being zero and in that \( a, b \) are such that the right hand side is zero, i.e., \( a_1 = a_6 = b_2 = b_4 = 1 \) and the others are zero.

\[
\begin{pmatrix}
0 & -1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_8 \\
\alpha_1 0 \\
\beta_8 \\
\beta_1 0
\end{pmatrix}
= \begin{pmatrix}
a_8 \\
a_{10} + 1 \\
b_8 \\
b_{10}
\end{pmatrix}.
\]

Again, all \( \alpha, \beta \) are zero and \( a_8 = b_8 = b_{10} = 0 \) while \( a_{10} = -1 \).

The other blocks (6 \( \times \) 6 and two 1 \( \times \) 1 blocks) have nonzero determinant and hence unique solution for whatever right hand side, in particular for the simplest: \( a = b = 0 \). The block equations read:

\[
\begin{pmatrix}
0 & -1 & 0 & -1 & 0 & 0 \\
2 & 0 & -2 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 2 & 0 & -2 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha_5 \\
\alpha_7 \\
\alpha_9 \\
\beta_5 \\
\beta_7 \\
\beta_9
\end{pmatrix}
= \begin{pmatrix}
a_5 + 2/3 \\
a_7 \\
a_9 + 1/3 \\
b_5 \\
b_7 \\
b_9
\end{pmatrix}.
\]

having solution for any \( a, b \) which allows us to choose \( a = b = 0 \) and adjust \( \alpha, \beta \) accordingly. The remaining two blocks affect \( a_3, b_3, \alpha_3, \alpha_4, \beta_5, \beta_7 \),
\( \beta_3 \) and is solved taking all of them to be zero. So we obtained: 
\[ F_3 = (u^3 + uv^2 - vz^2, v^3 + vu^2) \] and 
\[ (h_1, h_2)_3 = (-\frac{2}{9}uvz, -\frac{4}{9}u^2z - \frac{5}{9}v^2z). \]

Back to the equations, we rewrite \( z \) as \( \mu \) and get:

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = \begin{pmatrix}
\mu & -1 - \mu^2 \\
1 & \mu
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix} + (u^2 + v^2) \begin{pmatrix}
u \\
v
\end{pmatrix} + O(4)
\]

where we see that the quadratic term vanishes in a Hopf bifurcation.